UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Homogeneous descriptions and families of homogeneous structures

Descripciones homogéneas y familias de estructuras homogéneas

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

José Luis Carmona Jiménez

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Programa de doctorado en investigación matemática-D9B3

UNIVERSIDAD COMPLUTENSE MADRID

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A mi madre.

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Resumen

Los espacios homogéneos son, por su sencillez, los objetos favoritos de geómetras y físicos. Las propiedades locales, en muchos casos, se convierten en globales. Sin lugar a dudas, son los espacios más estudiados y dan lugar a los primeros ejemplos en muchas nuevas teorías. Por ejemplo, los espacios Eucídeos, las esferas, los espacios hiperbólicos o los grupos de Lie son espacios homogéneos.

Los espacios homogéneos son variedades diferenciables donde hay una acción transitiva, es decir, un grupo de transformaciones globales de manera que para cualquier par de puntos existe una transformación que envía uno en el otro. Utilizando esas transformaciones, bajo ciertas condiciones, podemos transportar cualquier tensor a todos los puntos de la variedad, como por ejemplo, un tensor métrico, o un tensor simpléctico. En particular, si existe una métrica diremos que el espacio homogéneos es Riemanniano.

El Teorema de Ambrose-Singer caracteriza los espacios homogéneo Riemannianos a través de la existencia de un tensor invariante que satisface un sistema de ecuaciones covariantes. Este tensor se llama estructura homogénea y el Teorema de Ambrose-Singer es la primera pieza del programa iniciado por Tricerri-Vanhecke y que estudia las variedades homogéneas Riemannianas a través del tensor estructura homogénea.

Este programa establece los siguientes principios: Caracterizar un espacio homogéneos específico es conocer todos los tensores estructura homogénea; Se pueden utilizar las estructuras homogéneas para diferenciar acciones transitivas diferentes.

En esta tesis el objeto de estudio son los espacios homogéneos y las técnicas que aplicaremos son algunas que surgen del programa de Tricerri-Vanhecke. Describir todas las estructuras homogéneas de un espacio homogéneo específico, en general, es un problema difícil. De hecho, no son conocidas todas las estructuras homogéneas de la mayoría de los espacios homogéneos Riemannianos. En primer lugar, caracterizamos todas las estructuras homogéneas del espacio hiperbólico complejo. A continuación, examinamos el proceso de reducción de estructuras homogéneas cuando la fibra de la reducción es unidimensional.

Numerosas generalizaciones del Teorema de Ambrose-Singer y el programa de Tricerri-Vanhecke se han ido descubriendo a lo largo de los años. Sin embargo, todas ellas tenían dos

características comunes: debe existir una métrica y se aplican a acciones transitivas. Siguiendo esa filosofía, la segunda parte de la tesis tiene como objetivo debilitar esas dos condiciones.

En espacios homogéneos reductivos, nosotros generalizamos el Teorema de Ambrose-Singer con presencia de un conjunto finito de tensores invariantes y sin ser necesario que una métrica esté presente. Damos una definición más general de estructura homogénea y nuevos tensores toman papel en la teoría más general. Finalmente, aplicamos estos resultados en los espacios homogéneos simplécticos.

En acciones no transitivas, nos centramos en variedades Riemannianas de cohomogeneidad uno, es decir, hay un grupo de transformaciones actuando en la variedad y las órbitas de la acción son hipersuperficies en la variedad. Este es el caso más cercano a una acción transitiva, sin ser transitiva. En este marco de trabajo, probamos que la existencia de una acción de cohomogeneidad uno en una variedad Riemanniana es equivalente a la existencia de un tensor que satisface un sistema de ecuaciones covariantes. Después, aplicamos esto para estudiar las acciones de cohomogeneidad uno en el espacio Euclídeo y el espacio hiperbólico real.

Abstract

Homogeneous spaces are, due to their simplicity, the favorite objects of study for geometers and physicists. Local properties often extend to global ones. Undoubtedly, these spaces are among the most extensively researched and give rise to the first examples of many new theories. These include Euclidean spaces, spheres, hyperbolic spaces, and Lie groups, all classified as homogeneous spaces.

Homogeneous spaces are differentiable manifolds where there is a transitive action of a Lie group. That is, a group of global transformations such that for any two points, there exists a transformation that sends one point to the other. Under suitable conditions and applying those transformations, we can transport any tensor we have at one point to another point, for example, a metric or a symplectic tensor. In particular, if a metric is present, the homogeneous space is called Riemannian.

The Ambrose-Singer Theorem characterizes Riemannian homogeneous spaces via the existence of an invariant tensor that satisfies a system of covariant equations. This tensor is called a homogeneous structure and the Ambrose-Singer Theorem is the cornerstone of the program initiated by Tricerri-Vanhecke that studies Riemannian homogeneous manifolds through their homogeneous structures.

This program establishes the following principles: To characterize a specific Riemannian homogeneous space is to know all its homogeneous structures; Homogeneous structures can be used to differentiate transitive actions.

In this thesis, the object of study is homogeneous spaces, and we apply techniques derived from the Tricerri-Vanhecke program. Describing all homogeneous structures of a specific homogeneous space is, in general, a challenging problem. In fact, many homogeneous structures of most Riemannian homogeneous spaces remain unknown. First, we characterize all the homogeneous structures of the complex hyperbolic space. Afterwards, we examine the process of reduction of homogeneous structures when the fibre of the reduction is one-dimensional.

Many generalizations of the Ambrose-Singer Theorem and the Tricerri-Vanhecke program have been discovered over the years. However, all of them share two common characteristics:

There must exist a metric tensor and they only apply to transitive actions. Following that philosophy, the second part of this thesis aims at tackling and weakening those two conditions.

In reductive homogeneous spaces, we generalize the Ambrose-Singer Theorem with the presence of a finite set of invariant tensors, eliminating the requirement of a metric. We provide a broader definition of homogeneous structure, and new tensors assume significance in the overarching theory. Finally, we apply those results to symplectic homogeneous spaces.

In non-transitive actions, we focus on Riemannian cohomogeneity one manifolds. That is, there is a group of transformations acting on the manifold in such a way that the orbits of the action are hypersurfaces. This is the closest case to a transitive action, without being transitive. In this framework, we prove that the existence of a Riemannian cohomogeneity one action is equivalent to the existence of a tensor satisfying a system of covariant equations. Then, we apply this to study cohomogeneity one actions on Euclidean spaces and real hyperbolic spaces.

Introduction

There is a fascination with homogeneous manifolds among geometers and physicists due to their elegance and the insights they offer into the interaction between symmetry and geometry. Moreover, good geometric examples are often homogeneous. Certainly, Euclidean spaces, Lie groups, symmetric spaces, spheres, hyperbolic spaces, projective spaces, Grassmannians, among others, exemplify homogeneous manifolds. Coupled with the characteristic of uniformity at every point, this elevates homogeneous manifolds as an ideal ground for testing conjectures or, conversely, discovering counterexamples to assertions. Indeed, computations are simplified, allowing us to infer statements from local to global. This renders differential geometry a domain where homogeneous manifolds serve as a preferred subject of study. This establishes them as a fundamental subject in the study of geometry, particularly concerning Lie groups and their actions.

This thesis is focused on the interplay of reductive homogeneous manifolds and Ambrose-Singer connections. These two topics are connected via the well-known Ambrose-Singer Theorem [AS58] and its generalizations [GO92; Kir80; Luj14]. The Ambrose-Singer connections $\tilde{\nabla}$ are affine connections satisfying,

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0,$$

where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion of $\tilde{\nabla}$. In the special case where (M,g) is a Riemannian manifold and $\tilde{\nabla}$ is metric, these equations can be equivalently expressed as:

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}S = 0,$$

where $S = \nabla - \tilde{\nabla}$ and ∇ denotes the Levi-Civita connection. The tensor S is called *the Riemannian homogeneous structure*, playing a central role in the narrative of AS-connections.

Ambrose-Singer Theorem ([AS58, p. 656])**.** *Let (M,g) be a connected and simply-connected complete Riemannian manifold. Then, the following statements are equivalent:*

1. *The manifold M is Riemannian homogeneous.*

2. *The manifold M admits a linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0, \quad \tilde{\nabla} g = 0, \tag{AS}$$

where R is the curvature tensor of the Levi-Civita connection ∇ and $S = \nabla - \tilde{\nabla}$.

Under the specified topological conditions, the presence of a homogeneous structure implies that the manifold is Riemannian homogeneous, and vice versa. In other words, the geometric characteristics of homogeneity can be interpreted via the equations involving covariant derivatives of a connection. Now, key questions arise: Is there a method to discern whether a Riemannian manifold is homogeneous? What essential information is required to distinguish among different homogeneous manifolds?

These inquiries prompted the initiation of a research program by Tricerri and Vanhecke in [TV83], which is called in literature as Riemannian homogeneous structures. The aim of this program is to thoroughly investigate homogeneous and locally homogeneous Riemannian manifolds, with a specific focus on the analysis of Ambrose-Singer connections and Riemannian homogeneous structures.

This program relies on two principles: To characterize a specific Riemannian homogeneous space is to know all its Riemannian homogeneous structures; Homogeneous structures can be used to differentiate transitive actions.

In the first principle, the existence of any homogeneous structures ensures that, for any two points, there exist a local isometry sending one point to the other which is affine with respect to the Ambrose-Singer connection. However, it is not always globally homogeneous. If we relax the topological conditions:

Theorem ([Tri92, p. 413, Thm. 2.1])**.** *A connected Riemannian manifold (M, g) is locally homogeneous if and only if there exists a linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0, \quad \tilde{\nabla} g = 0,$$

where R is the curvature tensor of the Levi-Civita connection ∇ and $S = \nabla - \tilde{\nabla}$.

That is, the existence of solutions to (AS) implies the existence of an isometric transitive action, and conversely. Thus, by scrutinizing all homogeneous structures we describe all transitive actions.

In the second principle, let V be the tangent space at one point, and consider the space of homogeneous structures,

$$\mathcal{S}(V) = \Big\{ S \in V^* \otimes V^* \wedge V^* : \, g(S_X Y, Z) + g(S_X Z, Y) = 0 \Big\}.$$

Since,

$$\tilde{\nabla}S = 0,$$

we have that S is invariant by parallel transport along closed curves. Symbolically,

$$\tilde{R}_{XY} \cdot S = 0$$

where $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$. Specifically, $\tilde{R}_{XY} \subset \mathrm{O}(n)$; thus, pointwise, by decomposing the module $\mathcal{S}(V)$ into $\mathrm{O}(n)$-irreducible submodules, we derive necessary conditions for two distinct homogeneous structures to be isomorphic. In simpler terms, the homogeneous structure helps to discern the essential conditions required for two Riemannian homogeneous manifolds to be identical under the same transitive action.

The Ambrose-Singer Theorem was extended to any Riemannian homogeneous manifold possessing a finite set of invariant tensors, as detailed in [Kir80]. Subsequently, the first generalization of the Ambrose-Singer Theorem to pseudo-Riemannian manifolds was demonstrated.

Theorem ([GO92])**.** *Let (M,g) be a connected and simply-connected pseudo-Riemannian manifold. Then, the following statements are equivalent:*

1. *The manifold M is* ***reductive*** *pseudo-Riemannian homogeneous.*
2. *The manifold M admits a* ***complete*** *linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}g = 0,$$

where R is the curvature tensor of the Levi-Civita connection ∇^{LC} and $S = \nabla^{LC} - \tilde{\nabla}$.

This generalization introduces novel concepts: Ambrose-Singer connections apply to reductive pseudo-Riemannian homogeneous manifolds; The requirement for completeness shifts from the Levi-Civita connection to the geodesic completeness of the Ambrose-Singer connection.

Any homogeneous manifold M, non necessarily Riemannian, is equal to G/H where G is the Lie group action and H is the isotropy group at one point. Under these conditions, it is reductive if, for the Lie algebra $\mathfrak{g}$ of G, there exists a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-subspace. It is worth noting that the majority of homogeneous manifolds fall into the reductive category, rendering this condition non restrictive. Specifically, every Riemannian homogeneous manifold is inherently reductive [CC19, p. 36].

Summarizing, the Tricerri-Vanhecke program could be adapted for pseudo-Riemannian homogeneous manifolds incorporating additional invariant tensors. In particular, if we decompose the space of homogeneous structures $\mathcal{S}(V)$ into the different Riemannian structure groups stabilizing these invariant tensors, then we get necessary conditions for these structures to be isomorphic in that underlying geometry. For example, this can be applied to manifolds that are: pseudo-hermitian, see [AG88]; para-hermitian, see [GO92]; pseudo-quaternion Kähler, see [CGS06]; para-quaternion Kähler, see [CC19]; almost contact metric, see [Fin98; CC19].

Outset

The first goal of this thesis is to give a quick overview and introduction to the Ambrose-Singer Theorem and its generalizations throughout the history together with the research program of Tricerri-Vanhecke and its applications.

In the first principle, we outlined that to describe all transitive actions of a Riemannian manifold (i. e., all the descriptions as $M = G/H$) it is equivalent to scrutinize all homogeneous structures. Surprisingly, comprehensive lists detailing all homogeneous descriptions of homogeneous spaces are still unknown in many cases. Furthermore, even if all transitive actions on a homogeneous space are known, questions about the geometry of the Ambrose-Singer connections or homogeneous structures are still unsolved in most cases. Some exceptions are: the Heisenberg group, see [TV83]; the spheres, see [AHL23]; the real hyperbolic space, see [CGS09; CGS13]. The second goal of this thesis is to scrutinize the Kähler homogeneous structures of the complex hyperbolic space.

It is remarkable that, in the situation where there is an invariant map between two homogeneous manifolds of different dimensions, the relationship of the homogeneous structures of both manifolds is mostly unknown. An example of this is the reduction procedure of homogeneous structures, which was first introduced in [CL15]. In particular, the authors reduce from almost contact metric homogeneous structures to almost pseudo-hermitian homogeneous structures. The third goal is to examine the converse reduction, that is, from almost pseudo-hermitian homogeneous structures to almost contact metric homogeneous structures.

Many generalizations of the Ambrose-Singer Theorem and the Tricerri-Vanhecke program have been discovered over the years. However, all of them share two common characteristics: There must exist a metric tensor and they only apply to transitive actions. On the one hand, if none invariant tensor is present in the manifold, we have

Theorem ([KN69, Thm. 2.8])**.** *Let M be a connected and simply-connected manifold. Then, the following assertions are equivalent:*

- *The manifold M is reductive homogeneous.*
- *The manifold M admits a complete linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0,$$

where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion tensor fields of the connection $\tilde{\nabla}$.

The fourth goal is to proof an Ambrose-Singer Theorem for general homogeneous manifolds equipped with invariant tensors and apply this result to start a Tricerri-Vanhecke program for non-metric homogeneous manifolds, for example, symplectic homogeneous manifolds.

On the other hand, if the action is not transitive, then this fruitful philosophy is unknown. The fifth goal is to be able to extrapolate this program to non-transitive and proper Riemannian actions when the principal orbits are hypersurfaces. It is desirable to characterize these actions in terms of some tensors as in Ambrose-Singer Theorem, to obtain classifications and to give explicit examples of these objects.

Outline

Chapter 1 provides an introduction to reductive homogeneous manifolds and exhibits the history of the Ambrose-Singer Theorem. The Tricerri and Vanhecke program, [TV83], is presented with examples at the end of the chapter.

Chapter 2 describes the holonomy algebras of all canonical connections and their action on complex hyperbolic spaces $\mathbb{C}\mathrm{H}(n)$ in all dimensions ($n \in \mathbb{N}$). This thorough investigation yields a formula for all Kähler homogeneous structures on complex hyperbolic spaces. Kähler homogeneous structures can be decomposed into the different Tricerri-Vanhecke (or Abbena-Garbiero) orthogonal and irreducible $\mathrm{U}(n)$-submodules. Thus, we characterize the holonomy of the canonical connection in terms of the projections of the homogeneous structures to the different submodules.

Chapter 3 studies the reduction procedure applied to pseudo-Kähler manifolds by a one dimensional Lie group acting by isometries and preserving the complex tensor. We endow the quotient manifold with an almost contact metric structure. We use this fact to connect pseudo-Kähler homogeneous structures with almost contact metric homogeneous structures. This relation will have consequences in the class of the almost contact manifold. Indeed, if we choose a pseudo-Kähler homogeneous structure of linear type, then the reduced almost contact homogeneous structure is of linear type and the reduced manifold is of type $\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_{12}$ in the Chinea-González classification [CG90].

Chapter 4 generalizes the Ambrose-Singer Theorem for general homogeneous manifolds, that is, the main result of this chapter provides a characterization of reductive homogeneous spaces equipped with some geometric structure (not necessarily pseudo-Riemannian) in terms of the existence of a certain connection. We relax the conditions in this theorem and prove a characterization of reductive locally homogeneous manifolds. We connect these two viewpoints with infinitesimal models, the Nomizu construction and the transvection construction and give coherent definitions of isomorphisms for all these classes.

Chapter 5 applies the results of Ch. 4 to symplectic geometry. We can classify reductive locally homogeneous symplectic manifolds by the existence of one Ambrose-Singer connection whose torsion belongs to some of the irreducible $\mathrm{Sp}(2n)$-submodules of the torsion-like tensorial space. Additionally, if we have a Fedosov manifold, we can consider the difference tensor between the Fedosov connection and the Ambrose-Singer connection. It is a Fedosov homogeneous structure. Thus, with an analogous procedure to the second principle mentioned above, but with a non-metric perspective. The action of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ decomposes the space of Fedosov homogeneous structures in irreducible components. The dimension of one of these components grows linearly with the dimension of the manifold. These homogeneous structures are called of linear type and we describe them in the last section.

Chapter 6 characterizes proper Riemannian actions when the principal orbits are hypersurfaces by the existence of a linear connection satisfying a set of covariant equations. We apply these results to give classifications of cohomogeneity one foliations and give explicit examples of these objects in the Euclidean space and the real hyperbolic space.

Outcome

The results of this thesis are available at:

[CC20] José Luis Carmona Jiménez and Marco Castrillón López. “Reduction of Homogeneous pseudo-Kähler Structures by One-Dimensional Fibers”. *Axioms* **9**:(3) (2020). DOI: 10.3390/axioms9030094.

[CC22a] José Luis Carmona Jiménez and Marco Castrillón López. “The Ambrose-Singer Theorem for General Homogeneous Manifolds with Applications to Symplectic Geometry”. *Mediterranean Journal of Mathematics* **19**: (2022), p. 280. DOI: 10.1007/s00009-022-02197-x.

[CC22b] José Luis Carmona Jiménez and Marco Castrillón López. “The homogeneous holonomies of complex hyperbolic space”. *Annals of Global Analysis and Geometry* **62**: (2022), pp. 391–411. DOI: 10.1007/s10455-022-09852-2.

[CCD23] José Luis Carmona Jiménez, Marco Castrillón López, and José Carlos Díaz-Ramos. “The Ambrose-Singer theorem for cohomogeneity one manifolds” (2023). arXiv: 2312.16934.

They are organized as follows. The papers [CC22b] and [CC20] are devoted to study pseudo-Riemannian homogeneous manifold. They collect the results of Chapter 2 and Chapter 3, respectively. The paper [CC22a] examines the generalization of the Ambrose-Singer Theorem to non-metric homogeneous manifolds. We split these results in Chapter 4 and Chapter 5. The manuscript [CCD23] extends the Tricerri-Vanhecke program to non-transitive actions when the orbits are hypersufaces. Then, it collects the results of Chapter 6.

Outlook

There are diverse future lines of research related to the results of this thesis:

All the homogeneous descriptions of a manifold: In [CGS13], the authors described all the homogeneous geometries of the real hyperbolic space. In Ch. 2, we show all the homogeneous geometries of the complex hyperbolic space.

> **Open problem 1:** Which are all the homogeneous descriptions of the quaternionic hyperbolic space, real or complex projective spaces, . . . ?

A definition for morphism of AS-manifolds: In Ch. 4, we prove that the class of an AS-manifold is larger than the class of homogeneous reductive manifolds. Moreover, in that chapter we define an isomorphism of AS-manifolds that generates a bijective correspondence with isomorphisms of infinitesimal models, the Nomizu construction and the transvection construction. The reduction morphism between pseudo-Riemannian manifolds is shown in Ch. 3.

> **Open problem 2:** Find a definition of morphism of AS-manifolds and morphism of infinitesimal models such that these two definitions fit into each other.

In particular, this definition enables us to find a way to connect the homogeneous structure of an intrinsically homogeneous submanifold with the homogeneous structure of the total homogeneous space. In this line of research not even a definition of morphism of homogeneous reductive manifolds is given. Therefore, there is a long way to explore.

Study of invariant tensors in homogeneous manifolds: In Ch. 2, we show that the symmetric description of the complex hyperbolic space of dimension n is the unique description without an invariant vector field. Meanwhile, the description of the complex hyperbolic space as a Lie group is the unique description with $2n$ invariant vector fields. Obviously, if two homogeneous descriptions have different invariant tensor fields, then necessarily they have to be different. The question is,

> **Open problem 3:** Is it possible to use the AS-connections to describe the space of all invariant tensor fields of a Lie group action?

Tricerri-Vanhecke program for non-metric homogeneous structures: In Ch. 5, we start this program for symplectic manifolds. This can be replicated in other geometries such as complex, contact, multisymplectic, ... Or combine some of these geometries such as symplectic with complex. Moreover, there are some characteristic connections that are related to these tensors. It should be possible to describe the class of homogeneous structures associated with these tensors.

> **Open problem 4:** Let (M, ∇, K) be an AS-manifold with AS-connection $\tilde{\nabla}$. Which are the invariant classes by the Lie structure group given by K of the AS-tensors and homogeneous structures?

The description of symplectic non-Fedosov homogeneous structures of linear type: In Ch. 5, we show that there is a class of almost symplectic homogeneous structures of linear type. There is a subclass of symplectic homogeneous structures that we can endow with a Fedosov invariant connection. At the end of that chapter we describe Fedosov homogeneous structures of linear type. However, there is another non described subclass of almost symplectic (non Fedosov) homogeneous structures of linear type.

> **Open problem 5:** Give a complete description of almost symplectic homogeneous structures of linear type.

All the Riemannian cohomogeneity one actions on a manifold: In Ch. 2, we compute all Kähler homogeneous structures on the complex hyperbolic space. Following these ideas but with a perspective of cohomogeneity actions, we have that to compute all the (canonical) cohomogeneity one structures is equivalent to giving a list of cohomogeneity one actions.

> **Open problem 6:** Compute all (canonical) cohomogeneity one structures of the real hyperbolic space, complex hyperbolic space, spheres, euclidean spaces, ... ?

Chapter 1

Preliminaries

This chapter introduces all the necessary knowledge to face the problems concerning Ambrose-Singer connections and homogeneous manifolds in this thesis. All the results are summarized without proof, although they are referenced. Our goal in this chapter is to understand the classical Ambrose-Singer theorem. To achieve that, there are two main concepts. First, *linear connections* are principal connections in the frame bundle, or equivalently, vectorial connections in the tangent bundle. We start with the perspective of principal bundles and connect this point of view with vector bundles. Moreover, we introduce parallel transport, holonomy, curvature and torsion tensors, geodesics, completeness of connections, affine maps, etcetera. Second, a *reductive homogeneous manifold* is a homogeneous manifold that satisfies a decomposition in terms of Lie algebras. This implies the existence of an unique canonical connection such that the affine maps are the Lie group actions on the manifold. It follows that its covariant derivative makes its curvature and torsion parallel. Indeed, this canonical connection is an Ambrose-Singer connection.

In the third section, we present the Ambrose-Singer theorem and its generalizations throughout history either with a local or global homogeneity condition. After that, we discuss the principal research program introduced by Franco Tricerri and Lieven Vanhecke in [TV83] and introduce the homogeneous structure tensor. This project establishes some necessary conditions for different homogeneous structures to be isomorphic.

Throughout the text, we assume that all objects are differentiable, and manifolds are finite-dimensional. The procedures and results presented in this chapter can be found in [KN63], [KN69], [TV83], or [CC19]. Furthermore, we adapt our notation to these books.

1.1 Principal bundles

Definition 1.1.1. Let P and M be manifolds, and let G be a Lie group. A *principal bundle* $(\pi : P \longrightarrow M, G)$ (or $(P \longrightarrow M, G)$) is a surjective submersion $\pi\colon P \longrightarrow M$ together with a free and $\mathcal{C}^\infty$ right action of G on P that is transitive on the fibers of π.

We denote by $R_g(u) = u \cdot g$ the action on the right of G on P and the manifolds P, M and G are called the *total space*, the *base space* and the *structure group*, respectively. The preimage $\pi^{-1}(p)$ of a point $p \in M$ is the *fiber of* p. We recall that an action is said to be *free* if given $p_1 \in P$ with $u_1 \cdot g = u_1$, then $g = e$ is the neutral element of G. Additionally, it is said to be *transitive on the fibers* if for every $p \in P$ and two u_1, $u_2 \in \pi^{-1}(p)$, then there exists $g \in G$ such that $R_g(u_1) = u_2$.

We assume throughout that the reader is familiar with the definition, basic properties and constructions concerning *vector bundles* (see [Hus66, Ch. 3] or [Mic08, Ch. IV]). Although Def. 1.1.1 does not exhibit the existence of local trivializations, as the vector bundles definition does, it is an immediate consequence.

Proposition 1.1.2 ([CC19, p. 3, Prop. 1.1.2])**.** *Let $(\pi\colon P \longrightarrow M, G)$ be a principal bundle. Given a point $p \in M$, there exists a neighbourhood U of p and a smooth map $\sigma\colon U \longrightarrow \pi^{-1}(U)$ such that $\pi \circ \sigma = \mathrm{id}$. Moreover, the map*

$$\begin{aligned} \phi\colon U \times G &\longrightarrow \pi^{-1}(U) \\ (p,g) &\longmapsto \sigma(p) \cdot g \end{aligned}$$

is a diffeomorphism satisfying $\pi \circ \phi(p,g) = p$ and $\phi(p, g \cdot h) = \phi(p,g) \cdot h$.

From Prop. 1.1.2, we have the following identity in terms of dimensions,

$$\dim P = \dim M + \dim G.$$

A *homomorphism of principal bundles* $(P_1 \longrightarrow M_1, G_1)$ to $(P_2 \longrightarrow M_2, G_2)$ is a pair (Φ, ϕ) of a differentiable map $\Phi\colon P_1 \longrightarrow P_2$ and a homomorphism of Lie groups $\phi\colon G_1 \longrightarrow G_2$, making commutative the following diagram,

$$\begin{tikzcd} P_1 \times G_1 \arrow[r, "(\Phi,\phi)"] \arrow[d, "\cdot G_1"'] & P_2 \times G_2 \arrow[d, "\cdot G_2"] \\ P_1 \arrow[r, "\Phi"] & P_2 \end{tikzcd}$$

or equivalently, $\Phi(u \cdot g) = \Phi(u) \cdot \phi(g)$, for all $u \in P_1$ and $g \in G_1$. Moreover, if Φ is an embedding and ϕ a monomorphism, then (Φ, ϕ) is a *principal subbundle*. A special case of subbundles are *reductions*, that is, subbundles with $M_1 = M_2$.

The *frame bundle of* M, which we denote by $(\mathcal{L}(M) \longrightarrow M, \mathrm{GL}(n))$, is the main example of principal bundles. The aim of this section is to provide a detailed exposition of it. In particular, the next paragraph gives its precise construction.

Let M be a differentiable manifold of dimension n. We consider,

$$\mathcal{L}(M) = \Big\{ u = (p; u_1, \ldots, u_n) : \, p \in M, \; (u_1, \ldots, u_n) \text{ is a basis of } T_pM \Big\},$$

or equivalently,

$$\mathcal{L}(M) = \Big\{ u : \mathbb{R}^n \longrightarrow T_pM : \, p \in M, u \in \mathrm{Isom}(\mathbb{R}^n, T_pM) \Big\},$$

which can be endowed with the structure of a differentiable manifold such that the natural projection $\pi \colon \mathcal{L}(M) \longrightarrow M$ is a surjective submersion. Let now $\mathrm{GL}(n)$ be the Lie group of real invertible matrices of dimension n, or equivalently, the Lie group of linear isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. We define the action of the Lie group $\mathrm{GL}(n)$ on $\mathcal{L}(M)$ as follows,

$$\begin{aligned} \mathcal{L}(M) \times \mathrm{GL}(n) &\longrightarrow \mathcal{L}(M) \\ ((p;\, u_1, \ldots, u_n), A) &\longmapsto (p;\, u_1, \ldots, u_n) \cdot A = \left(p; \sum_{i=1}^{n} a_{i1} u_i, \ldots, \sum_{i=1}^{n} a_{in} u_i \right) \end{aligned}$$

where $A = (a_{ij}) \in \mathrm{GL}(n)$. Equivalently, we can define the same action as,

$$\begin{aligned} \mathcal{L}(M) \times \mathrm{GL}(n) &\longrightarrow \mathcal{L}(M) \\ (u, \varphi) &\longmapsto u \cdot \varphi = u \circ \varphi \end{aligned}$$

where $(\varphi \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n) \in \mathrm{GL}(n)$. Moreover, for all $p \in M$, the action is free and transitive on every set $\pi^{-1}(p)$. Let $f \colon M \longrightarrow M$ be a differentiable map. Then, we define the *natural lift* of f by,

$$\begin{aligned} \tilde{f} \colon \mathcal{L}(M_1) &\longrightarrow \mathcal{L}(M_2) \\ u &\longmapsto f_{*,\pi(u)} \circ u. \end{aligned} \tag{1.1}$$

In particular, $(\tilde{f}, \mathrm{Id})$ defines a homomorphism of principal bundles.

Let us now consider a principal bundle $(P \longrightarrow M, G)$ such that its structure group G acts linearly on the left on a vector space V of finite dimension. Then, an induced right action of G

on $P \times V$ is defined as

$$(u,\xi)\cdot g = (u\cdot g, g^{-1}\cdot \xi).$$

The quotient $E = P\times_G V = (P\times V)/G$ together with the projection $\pi_E\colon E \longrightarrow M$ such that $\pi_E([u,\xi]_G) = \pi(u)$ is a vector bundle called the *associated bundle* to $(P\longrightarrow M, G)$ with fiber V.

There exists a correspondence between sections $\phi\colon M\longrightarrow E$ of the associated bundle $(E\longrightarrow M, V)$ and G-equivariant maps $F^{\phi}\colon P\longrightarrow V$. The correspondence is given by

$$\phi(\pi(u)) = [u, F^{\phi}(u)]_G.$$

This function F^{ϕ} is called the *G-equivariant map* associated with ϕ.

We now show how to construct TM explicitly based on this perspective and the correspondence explained above. Let $(\mathcal{L}(M)\longrightarrow M, \mathrm{GL}(n))$ be the frame bundle of M and let $V = \mathbb{R}^n$. If we consider the natural action of $\mathrm{GL}(n)$ on V, then

$$\mathcal{L}(M)\times_{\mathrm{GL}(n)}\mathbb{R}^n$$

is canonically isomorphic to *the tangent bundle* by

$$\begin{aligned} \mathcal{L}(M)\times_{\mathrm{GL}(n)}\mathbb{R}^n &\longrightarrow TM\\ [u,\xi]_{\mathrm{GL}(n)} &\longmapsto u(\xi), \end{aligned}$$

the inverse of which is

$$\begin{aligned} TM &\longrightarrow \mathcal{L}(M)\times_{\mathrm{GL}(n)}\mathbb{R}^n\\ \eta &\longmapsto [u,\xi]_{\mathrm{GL}(n)} \end{aligned}$$

where u is a reference such that $u(\xi)=\eta$.

Let $X_{\pi(u)} = [u,\xi]_{\mathrm{GL}(n)} = u(\xi)$ be a vector field in $TM\cong \mathcal{L}(M)\times_{GL(n)}\mathbb{R}^n$, then there exists a G-equivariant map $F\colon \mathcal{L}(M)\longrightarrow\mathbb{R}^n$ such that $F(u\cdot g) = g^{-1}\cdot F(u)$. This map is given by $F(u) = u^{-1}\left(X_{\pi(u)}\right)$. Conversely, for any G-equivariant map $F\colon \mathcal{L}(M)\longrightarrow\mathbb{R}^n$, there exists a vector field $X_{\pi(u)} = [u, F(u)]_{\mathrm{GL}(n)}$.

In general, if we take $V = (\otimes^r\mathbb{R}^n)\otimes(\otimes^s(\mathbb{R}^n)^*)$ and the natural action of $\mathrm{GL}(n)$ on V, then

$$\mathcal{L}(M)\times_{\mathrm{GL}(n)}((\otimes^r\mathbb{R}^n)\otimes(\otimes^s(\mathbb{R}^n)^*))$$

is the vector bundle $\mathcal{T}^r_s(M)$ of (r,s)-tensor fields. Actually, there exists a correspondence between tensor fields of type (r,s) and G-equivariant functions $F\colon \mathcal{L}(M)\longrightarrow(\otimes^r\mathbb{R}^n)\otimes(\otimes^s(\mathbb{R}^n)^*)$.

1.1.1 Connections on Principal Bundles

Given $u \in P$ such that $\pi(u) = p$, we say that a vector in $B \in T_uP$ is *vertical* if $\pi_*(B) = 0$ or equivalently, B is in the tangent space of $\pi^{-1}(p)$. We call *vertical subspace* on u and *vertical distribution* to the sets $V_uP = \ker(\pi_* \colon T_uP \longrightarrow T_pM)$ and $VP = \ker(\pi_* : TP \longrightarrow TM)$, respectively.

Definition 1.1.3. A *connection (or principal connection)* Γ on a principal bundle $(P \longrightarrow M, G)$ is a *horizontal distribution* HP such that it is G-invariant,

$$(R_g)_*(H_uP) = H_{u\cdot g}P, \quad u \in P, g \in G,$$

and complementary to VP,

$$T_uP = H_uP \oplus V_uP, \quad u \in P.$$

Given $X_u \in T_uP$, we can decompose $X_u = X_u^h + X_u^v$, where $X_u^h \in H_uP$ and $X_u^v \in V_uP$ denote the *horizontal* and *vertical part* of X_u with respect to Γ, respectively. Let $\mathfrak{g}$ be the Lie algebra of G. The *fundamental vector field* associated with $A \in \mathfrak{g}$ is

$$A_u^* = \left.\frac{d}{dt}\right|_{t=0} u \cdot \exp(tA), \quad u \in P. \tag{1.2}$$

These vector fields satisfy that,

$$\begin{aligned} (R_g)_*(A^*) &= (Ad_{g^{-1}}(A))^*, \quad && g \in G, A \in \mathfrak{g}, \\ [A^*, B^*] &= [A, B]^*, && A, B \in \mathfrak{g}. \end{aligned}$$

For $u \in P$, the correspondence from $A \in \mathfrak{g}$ to A_u^* defines an isomorphism $\psi_u \colon \mathfrak{g} \longrightarrow V_uP$. We can define a 1-form ω on P with values in $\mathfrak{g}$ such that for every $X_u \in T_uP$, $\omega(X_u) = \psi_u^{-1}(X_u^v)$, that is, $\omega(X_u)$ is the unique element A such that $A_u^* = X_u^v$. This 1-form is called the *connection form* of Γ and satisfies,

$$\begin{aligned} &\omega(A^*) = A, && A \in \mathfrak{g}, \\ &\omega \text{ is } G\text{-equivariant} : (R_g)^*\omega = \mathrm{Ad}_{g^{-1}}\omega, \quad && g \in G. \end{aligned} \tag{1.3}$$

Conversely, for every 1-form on P with values in $\mathfrak{g}$ and satisfying (1.3), there exists an unique horizontal distribution HP defining a connection Γ and it is constructed by $H_uP = \ker(\omega_u)$.

Let (Φ, ϕ) be a homomorphism of principal bundles from $(P_1 \longrightarrow M_1, G_1)$ to $(P_2 \longrightarrow M_2, G_2)$ and two respective connections ω_1 (and horizontal distribution HP_1) and ω_2 (and horizontal distribution HP_2). We say that the homomorphism is *preserving the connections* if $\omega_2 \circ \Phi_* = \phi_* \circ \omega_1$ (or equivalently, $\Phi_*(HP_1) \subset HP_2$).

In particular, if (Φ, ϕ) is a reduction and ω_2 a connection on P_2, then, we say that ω_2 is *reducible* to P_1 if there exists a connection ω_1 such that the reduction sends ω_1 to ω_2.

Definition 1.1.4. Let Γ be a connection on $(P \longrightarrow M, G)$ and ω be the connection form. *The curvature form* Ω *of* Γ is the 2-form on P with values in $\mathfrak{g}$ given by, $\Omega(X,Y) = d\omega(X^h, Y^h)$.

Remark 1.1.5. The convention for the *exterior derivative* d is,

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$

where α is a 1-form and X, Y are vector fields.

The curvature form satisfies $(R_g)^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega$.

Theorem 1.1.6 (Second structure equation, [KN63, p. 77])**.** *Let* Ω *be the curvature form of* Γ *with connection form* ω. *Then,*

$$\Omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)],$$

where $[\cdot,\cdot]$ *is the Lie bracket of the Lie algebra* $\mathfrak{g}$ *and* X, $Y \in T_uP$, $u \in P$. *In particular, if* X, Y *are horizontal vector fields, then* $\Omega(X,Y) = -\omega([X,Y])$.

Note that, the curvature form is the obstruction for the horizontal distribution to be integrable. This is, $\Omega = 0$ if and only if HP is an integrable distribution.

Theorem 1.1.7 (Second Bianchi's identity, [KN63, p. 78])**.** *Let* Ω *be the curvature form of* Γ *with connection form* ω. *Then,*

$$d\Omega_u(X^h, Y^h, Z^h) = 0,$$

for all X, $Y \in T_uP$, $u \in P$.

1.1.2 Parallel transport and holonomy

Given a vector $X_p \in T_pM$, we define the *horizontal lift* of X_p to the point $u \in \pi^{-1}(p)$ to be the unique vector $\overline{X}_u \in H_uP$ such that $\pi_*(\overline{X}_u) = X_p$. Analogously, given a vector field $X \in \mathfrak{X}(M)$, we can define its horizontal lift as the unique horizontal vector field $\overline{X}$ in P with $\pi_*(\overline{X}) = X$. Let X, Y be two vector fields on M, then

$$\begin{aligned} &\overline{X} = (R_g)_*\overline{X}, && g \in G,\\ &\overline{(\lambda X + \mu Y)} = \lambda\overline{X} + \mu\overline{Y}, && \lambda, \mu \in \mathbb{R},\\ &[\overline{X}, \overline{Y}]^h = \overline{[X,Y]}. && \end{aligned}$$

Let γ be a piecewise differential curve on P. If its tangent vectors are horizontal, then we say that is *horizontal*. Therefore, given a piecewise differential curve $\tau\colon [0,1] \longrightarrow M$, $t \mapsto \tau_t$, on M and $u \in P$ with $\tau_0 = \pi(u)$, then there is an unique horizontal curve $\overline{\tau}^u$ on P, called the *horizontal lift* of τ with respect to the connection Γ, such that $\overline{\tau}^u_0 = u$ and $\pi(\overline{\tau}^u_t) = \tau_t$.

Definition 1.1.8. Let $\tau\colon [0,1] \longrightarrow M$ be a piecewise differential curve on M. We define the *parallel transport of* Γ *along* τ as the G-equivariant map,

$$\begin{aligned} ||_{\tau_0}^{\tau_1}(\tau,\Gamma)\colon \pi^{-1}(\tau_0) &\longrightarrow \pi^{-1}(\tau_1) \\ u &\longmapsto \overline{\tau}^u_1 \end{aligned}$$

for short we write $||_{\tau_0}^{\tau_1}(\tau)$ or $||_{\tau_0}^{\tau_1}$. We denote by $||(\Gamma)$ the set of all parallel transports of Γ along any piecewise differential curve on M.

For a deeper discussion of parallel transport, we refer the reader to [Mic08, Ch. IV].

Let $p \in M$ be a point and we denote by $C(p)$ and $C^0(p)$ the sets of all $\mathcal{C}^\infty$ loops and contractible $\mathcal{C}^\infty$ loops based at p, respectively.

Definition 1.1.9. Let Γ be a connection on a principal bundle $(P \longrightarrow M, G)$ and let $p \in M$. The *holonomy group* of Γ at p is

$$Hol(p,\Gamma) = \left\{ ||_p^p(\tau)\colon \pi^{-1}(p) \longrightarrow \pi^{-1}(p) : \tau \in C(p) \right\}$$

and the *restricted holonomy group* of Γ at p is

$$Hol^0(p,\Gamma) = \left\{ ||_p^p(\tau)\colon \pi^{-1}(p) \longrightarrow \pi^{-1}(p) : \tau \in C^0(p) \right\}.$$

For short, we denote these two by $Hol(p)$ and $Hol^0(p)$, respectively.

Remark 1.1.10. The holonomy group is a group with the composition of maps and the restricted holonomy group is a subgroup.

Let $u \in \pi^{-1}(p)$ be a fixed point. Then, we can consider the following homomorphisms of groups,

$$\begin{aligned} \Lambda_u\colon Hol(p) &\longrightarrow G \\ ||_p^p(\tau) &\longmapsto g \end{aligned}$$

such that, $\left(||_p^p(\tau)\right)(u) = u \cdot g$. The image of Λ_u is the subgroup $Hol(u)$ of G called the *holonomy group of* Γ *with base point* $u \in P$. With an analogous procedure, we define $Hol^0(u) = \Lambda_u(Hol^0(p))$, the *restricted holonomy group of* Γ *with base point* $u \in P$. If u is connected with v by a horizontal curve, then $Hol(u) = Hol(v)$.

Theorem 1.1.11 ([KN63, pp. 73-75]). *Let Γ be a connection on a principal bundle $(P \longrightarrow M, G)$ where M is connected. Let $Hol(u)$ and $Hol^0(u)$ be the holonomy and the restricted holonomy groups of Γ with base point $u \in P$, respectively. Then,*

1. *$Hol^0(u)$ is a Lie subgroup of G.*

2. *$Hol^0(u)$ is a normal subgroup of $Hol(u)$ and $Hol(u)/Hol^0(u)$ is countable.*

As consequence of these two, $Hol(u)$ is a Lie subgroup of G.

Let $u \in P$ be a fixed point. We can consider the set,

$$\mathcal{P}(u) = \left\{ v \in P : \left(||_{\tau_0}^{\tau_1}(\tau) \right)(u) = v \right\}.$$

It is straightforward to check that $(\mathcal{P}(u) \longrightarrow M, Hol(u))$ is a principal bundle, it is called the *holonomy bundle of Γ with base point $u \in P$*.

Let $(P \longrightarrow M, G)$ be a principal bundle, $u, v \in \pi^{-1}(p)$, and $g \in G$ such that $v = u \cdot g$. Then $\mathcal{P}(v) = \mathcal{P}(u) \cdot g$ and $Hol(v) = g^{-1} \cdot Hol(u) \cdot g$, that is, there is a homomorphism between different holonomy bundles.

Theorem 1.1.12 (Reduction theorem, [AS53], [KN63, pp. 83-85]). *Let Γ be a connection on a principal bundle $(P \longrightarrow M, G)$ and let $u \in P$. Then, $(\mathcal{P}(u) \longrightarrow M, Hol(u))$ is a reduction of $(P \longrightarrow M, G)$ and the connection Γ is also reducible.*

Theorem 1.1.13 (Holonomy theorem, [AS53], [KN63, pp. 89-90]). *Let Γ be a connection on a principal bundle $(P \longrightarrow M, G)$ and let $u \in P$. Let Ω be the curvature form of Γ and $(\mathcal{P}(u) \longrightarrow M, Hol(u))$ be the holonomy bundle. Then, the Lie algebra of $Hol(u)$ is the subalgebra $\mathfrak{hol}(u) \subset \mathfrak{g}$ spanned by all elements of the form $\Omega_v(X,Y)$, X, $Y \in T_vP$, with $v \in \mathcal{P}(u)$.*

1.1.3 Linear connections

Linear connections are intimately related to vectorial connections in the tangent bundle, principal connections in the frame bundle or covariant derivatives in the tangent bundle. The purpose of this section is to connect these three points of view of linear connections.

Let $(\pi_E \colon E \longrightarrow M, V)$ be a vector bundle. The *vertical distribution* is $VE = \ker((\pi_E)_*)$, that is, the tangent space to the fibers at every point of E. A *connection on a vector bundle* Γ_V (see [Lee09, Def. 12.12]) is a distribution HE on E such that,

- it is complementary to the vertical distribution, that is, $TE = HE \oplus VE$,

- for every point $(p,v)\in U\times V$ (a local trivialization), HE is invariant by multiplication,
$$(\mu_r)_* HE_{(p,v)} = HE_{(p,r\cdot v)}$$
where $\mu_r(p,v)=(p,r\cdot v)$.

The subbundle HE is called the *horizontal distribution*.

Let Γ be a principal connection on $(P\longrightarrow M,G)$, let V be a vector space such that G acts linearly on the left on V and let $(\pi_E\colon E\longrightarrow M,V)$ be the associated bundle. For $\xi\in V$ a fixed point, we consider the natural projection,

$$\begin{aligned}\Phi_\xi\colon P &\longrightarrow E = P\times_G V\\ u &\longmapsto [u,\xi]_G.\end{aligned}$$

Then, we consider the *horizontal distribution* $HE=\cup_{\xi\in V}(\Phi_\xi)_*(HP)$ associated with Γ with decomposition $T_wE=H_wE\oplus V_wE$ for every $w\in E$. This horizontal distribution on E defines a connection Γ_V on the vector bundle $(E\longrightarrow M,V)$.

Parallel transport for vector bundles has an analogous definition to that of principal bundles; a classical reference here is [Mic08, Ch. IV]. We denote the parallel transport of Γ_V a long a curve $\tau\colon[0,1]\longrightarrow M$ as $||_{\tau_0}^{\tau_1}(\tau)$, or $||_{\tau_0}^{\tau_1}$.

Remark 1.1.14. [KN63, p. 87] Summarising, given a principal connection Γ on a principal bundle $(P\longrightarrow M,G)$ and let V be a vector space such that G acts linearly on the left, then in the associated bundle $(E=P\times_G V\longrightarrow M,V)$ there exists an unique connection Γ_V such that the following diagram is commutative

$$\begin{array}{ccc} \pi^{-1}(\tau_0) & \xrightarrow{||_{\tau_0}^{\tau_1}(\tau,\Gamma)} & \pi^{-1}(\tau_1)\\ {\scriptstyle\Phi_\xi}\downarrow & & \downarrow{\scriptstyle\Phi_\xi}\\ \pi_E^{-1}(\tau_0) & \xrightarrow{||_{\tau_0}^{\tau_1}(\tau,\Gamma_V)} & \pi_E^{-1}(\tau_1)\end{array},\qquad \xi\in V.$$

Definition 1.1.15. A *linear connection* is a principal connection on the frame bundle.

In particular, every linear connection induces a connection and parallel transport on the tangent bundle. Conversely, given a connection Γ_V in the tangent bundle, we can induce a parallel transport given by: if $||_{\tau_0}^{\tau_1}\colon T_{\tau_0}M\longrightarrow T_{\tau_1}M$ is the parallel transport a long τ, then $||_{\tau_0}^{\tau_1}\colon \pi^{-1}(\tau_0)\longrightarrow\pi^{-1}(\tau_1)$ such that $u\mapsto ||_{\tau_0}^{\tau_1}\circ u$.

Definition 1.1.16. A covariant derivative ∇ on a vector bundle $(E \longrightarrow M, V)$ is a map,

$$\begin{aligned} \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, r) &\longmapsto \nabla_X r \end{aligned}$$

satisfying,

1. $\nabla_X(\lambda r + \mu s) = \lambda \nabla_X r + \mu \nabla_X s, \quad \lambda, \mu \in \mathbb{R}$.
2. $\nabla_{fX} r = f\, \nabla_X r, \quad f \in \mathcal{C}^\infty(M)$.
3. $\nabla_X(fr) = f(p)\nabla_X r + X(f)r, \quad f \in \mathcal{C}^\infty(M)$.
4. $\nabla_{X+Y} r = \nabla_X r + \nabla_Y r$.

where X, $Y \in \mathfrak{X}(M)$ and $r, s \in \Gamma(E)$.

The following Theorem states that there exists a correspondence between connections on vector bundles and covariant derivatives on vectors bundles. From now on, we denote connections and covariant derivatives on vector bundles with the same character ∇.

Theorem 1.1.17 ([Lee09, p. 520, Thm. 12.32])**.** *Let Γ be a connection on a vector bundle $(E \longrightarrow M, V)$. Then, there exists an unique covariant derivative ∇ such that*

$$\begin{aligned} \nabla_X \phi \colon M &\longrightarrow E \\ p &\longmapsto \nabla_{X_p} \phi \end{aligned}$$

where $\phi \in \Gamma(E)$ and

$$\nabla_{X_p} \phi = \lim_{t \longrightarrow 0} \frac{1}{t} \left(||_{\tau_t}^{\tau_0}(\tau, \Gamma)(\phi(\tau_t)) - \phi(\tau_0) \right)$$

with τ_t is a curve on M such that $\tau_0 = p$ and $X_p = \dot{\tau}_0$. Conversely, given a covariant derivative ∇ on a vector bundle $(E \longrightarrow M, V)$, then there exists an unique connection Γ such that

$$H_w E = \{\phi_{*,p}(X) \in T_w E : \ \phi \in \Gamma(E),\, \phi(p) = w,\, (\nabla_X \phi)_p = 0,\, p \in M\}.$$

The relation between principal connections and covariant derivative on its associated vector bundles is given below.

Proposition 1.1.18 ([KN63, p. 116, Prop. 1.3])**.** *Let Γ be a connection on $(\mathcal{L}(M) \longrightarrow M, \mathrm{GL}(n))$ and V be a vector space such that* $\mathrm{GL}(n)$ *acts linearly on the left. Let ϕ be a section of the* *associated vector bundle $(E \longrightarrow M, V)$ and let ∇ be the covariant derivative associated with Γ_V (the associated vectorial connection). Then, for any $X \in \mathfrak{X}(M)$, the* $\mathrm{GL}(n)$*-equivariant function $F^{\nabla_X \phi}$ associated with $\nabla_X \phi$ is*

$$F^{\nabla_X \phi} = \overline{X}(F^{\phi})$$

where $\overline{X}$ is the horizontal lift of X to P and F^{ϕ} is the $\mathrm{GL}(n)$*-equivariant funtion associated with ϕ.*

Remark 1.1.19. Prop. 1.1.18 is in general true for arbitrary principal bundles and their associated vector bundles.

Now, we define some geometric tensors that live only in the frame bundle. First, the *canonical form (or contact form)* θ of $\mathcal{L}(M)$ is the $\mathbb{R}^n$-valued 1-form given by

$$\theta(X_u) = u^{-1}(\pi_*(X_u)), \qquad X_u \in T_u\mathcal{L}(M), u \in \mathcal{L}(M).$$

It satisfies $R_g^*\theta = g^{-1} \cdot \theta$ for all $g \in \mathrm{GL}(n)$. Second, for every $\eta \in \mathbb{R}^n$, there exists an unique horizontal vector field, called *standard vector field associated with* η, $B(\eta)_u \in T_u\mathcal{L}(M)$ such that $\pi_*(B(\eta)) = u(\eta)$ for all $u \in \mathcal{L}(M)$. This vector field satisfies $\theta(B(\eta)) = \eta$, $(R_g)_*(B(\eta)) = B(g^{-1}\eta)$ for all $g \in \mathrm{GL}(n)$ and for any $A^* \in \mathfrak{gl}(n,\mathbb{R})$, $[A^*, B(\eta)] = B(A\eta)$. Additionally, the infinitesimal version of these natural lifts in $\mathcal{L}(M)$ depends on the canonical form.

Lemma 1.1.20 ([CC19, p. 36])**.** *Let X be a vector field on M. Then there exists an unique vector field $\tilde{X}$ on $\mathcal{L}(M)$ satisfying*

1. *$\tilde{X}$ is invariant under the right action of* $\mathrm{GL}(n)$.
2. *$\mathcal{L}_{\tilde{X}}\theta = 0$.*
3. *$\pi_*(\tilde{X}_u) = X_{\pi(u)}$, for all $u \in \mathcal{L}(M)$.*

We say that $\tilde{X}$ is the *natural lift* of X. And if $X_p = \frac{d}{dt}\big|_{t=0} f_t(p)$, then $\tilde{X}_u = \frac{d}{dt}\big|_{t=0}\tilde{f}_t(u)$, where $\tilde{f}$ is the natural lift of f_t to $\mathcal{L}(M)$.

Definition 1.1.21. Let Γ be a linear connection on $(\mathcal{L}(M) \longrightarrow M, G)$ and θ its canonical form. *The torsion form* Θ *of a linear connection* Γ is the 2-form on $\mathcal{L}(M)$ with values in $\mathbb{R}^n$ given by, $\Theta(X,Y) = d\theta(X^h, Y^h)$.

In particular, $R_g^*\Theta = g^{-1} \cdot \Theta$ for all $g \in \mathrm{GL}(n)$.

Theorem 1.1.22 (First structure equation, [KN63, p. 120])**.** *Let Θ be the torsion form of Γ with canonical form θ and connection form ω. Then,*

$$\Theta_u(X,Y) = d\theta(X,Y) + \omega(X)\cdot\theta(Y) - \omega(Y)\cdot\theta(X)$$

for all X, $Y \in T_u\mathcal{L}(M)$, $u \in \mathcal{L}(M)$. In particular, if X, Y are horizontal vector fields, then $\Theta(X,Y) = -\theta([X,Y])$.

The torsion form of a connection measures how horizontal vectors rotate with parallel transport along horizontal curves.

Theorem 1.1.23 (First Bianchi's identity, [KN63, p. 121])**.** *Let Θ be the torsion form of a linear connection Γ with connection form ω and canonical form θ. Then,*

$$\begin{aligned}(d\Theta)_u(X^h,Y^h,Z^h) &= +(\Omega\wedge\theta)_u(X,Y,Z)\\ &= \Omega_u(X,Y)\cdot\theta_u(Z) + \Omega_u(Z,X)\cdot\theta_u(Y) + \Omega_u(Y,Z)\cdot\theta_u(X),\end{aligned}$$

for all X, Y, $Z \in T_u\mathcal{L}(M)$, $u \in \mathcal{L}(M)$.

We define the *curvature tensor field* of ∇ as the $(1,3)$ tensor fields

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z),$$

and the *torsion field* of ∇ as the $(1,2)$-tensor field

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Proposition 1.1.24 ([KN63, p. 133])**.** *Let ∇ be a linear connection on M. Then,*

- *The* $\mathrm{GL}(n)$*-equivariant function associated with the torsion tensor field R of ∇ is*

$$\begin{aligned} r\colon \mathcal{L}(M) &\longrightarrow (\otimes^3(\mathbb{R}^n)^*)\otimes\mathbb{R}^n\\ u &\longmapsto \Big((\eta,\xi,\zeta)\mapsto \Omega_u\big(B(\eta)_u, B(\xi)_u\big)\cdot\zeta\Big).\end{aligned}$$

- *The* $\mathrm{GL}(n)$*-equivariant function associated with the torsion tensor field T of ∇ is*

$$\begin{aligned} t\colon \mathcal{L}(M) &\longrightarrow (\otimes^2(\mathbb{R}^n)^*)\otimes\mathbb{R}^n\\ u &\longmapsto \Big((\eta,\xi)\mapsto \Theta_u\big(B(\eta)_u, B(\xi)_u\big)\Big).\end{aligned}$$

A curve $\gamma\colon I \longrightarrow M$, where I is an open interval in $\mathbb{R}$, is called a *geodesic* if the vector field $X_{\gamma_{t_0}} = \dot{\gamma}_t$ is parallel along γ, or equivalently, if

$$\nabla_X X = 0.$$

Moreover, it is *complete* if $I = \mathbb{R}$. A *complete connection* is a linear connection whose geodesics are complete. In particular,

Proposition 1.1.25 ([KN63, Prop. 6.5])**.** *A linear connection is complete if and only if every standard vector field on $\mathcal{L}(M)$ is complete.*

Definition 1.1.26. Let (M_1, ∇_1) and (M_2, ∇_2) be two manifolds with two linear connections. We call $f\colon M_1 \longrightarrow M_2$ an *affine diffeomorphism* if it is a diffeomorphism and one of the following three happens

- The map $f_* : TM_1 \longrightarrow TM_2$ takes horizontal curves to horizontal curves.
- The map $f_* : TM_1 \longrightarrow TM_2$ satisfies $f_*((\nabla_1)_X Y) = (\nabla_2)_{f_* X} f_* Y$.
- The natural lift $\tilde{f}\colon \mathcal{L}(M_1) \longrightarrow \mathcal{L}(M_2)$ restricts to $\tilde{f}\colon \mathcal{P}(u_0) \longrightarrow \mathcal{P}(\tilde{f}(u_0))$.

In particular, affine maps preserve the curvature, the torsion and the geodesics.

1.1.4 Local coordinates

We express the canonical form θ and the connection form ω in terms of local coordinate systems. Let $\chi : V \subset \mathbb{R}^n \longrightarrow U \subset M$ be a surjective chart with local coordinate system $\chi = (x^1, \dots, x^n)$. We define,

$$\begin{aligned} \sigma\colon U &\longrightarrow \mathcal{L}(M) \\ p &\longmapsto \chi_{*,x} = \left(p; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \end{aligned}$$

where $x = \chi^{-1}(p)$. This map is a section of the frame bundle and we can define the trivialization given by,

$$\begin{aligned} \phi\colon U \times \mathrm{GL}(n) &\longrightarrow \pi^{-1}(U) \\ (p, \psi) = (p, A = (a_i^j)) &\longmapsto \chi_* \circ \psi = \left(\sigma(p); \sum_i a_1^i \frac{\partial}{\partial x_i}, \dots, \sum_i a_n^i \frac{\partial}{\partial x_i}\right). \end{aligned}$$

Then, the canonical form $\theta_{U\times \mathrm{GL}(n)} = \phi^*\theta$ has the expression in the coordinate system induced by χ (cf. [KN63, p. 140]),

$$\begin{aligned}\theta_{(p,\psi)} &= \psi^{-1}\circ\sigma(p)^{-1}\circ\pi_*\circ(\phi)_{*,(p,\psi)}\\ &= \psi^{-1}\circ\sigma(p)^{-1}\circ(\pi_U)_{*,(p,\psi)}\end{aligned}$$

where $\pi_U : U\times \mathrm{GL}(n)\longrightarrow U$ is the natural projection. Or equivalently, let $A\in \mathrm{GL}(n)$ such that $A = (a_i^j)$ and $A^{-1} = (b_i^j)$. Then,

$$\theta_{(p,A)} = \sum_{i=1}^{n}\theta^i_{(p,A)}e_i$$

where

$$\theta^i_{(p,A)} = \sum_{j=1}^{n} b^i_j dx^j.$$

Consider ω the connection form and let $\{E_i^j\}$ be a basis of $\mathfrak{g}$, where E_i^j is the zero matrix with a one only in the position (i,j) where i is the row and j is the column. Then, we can decompose,

$$\omega = \sum_{i,j=1}^{n}\omega^i_j E_i^j.$$

Then, we define the *components of the connection* Γ *(or the Christoffel symbols of* Γ*)* with respect to the coordinate system $\chi = (x^1,\ \dots\ ,x^n)$ as the maps $\Gamma^i_{kj}(p) = \omega^i_j\left(\sigma_*\left(\frac{\partial}{\partial x_k}\right)_p\right)$. According to this notation, the connection form $\omega_{U\times \mathrm{GL}(n)} = \phi^*\omega$ has the expression in the coordinate system χ (cf. [KN63, p. 140]),

$$\begin{aligned}\omega_{(p,\mathrm{id})}\left(\frac{\partial}{\partial x_k} + (E_r^s)^*\right) &= \sum_{i,j=1}^{n}\Gamma^i_{kj}E_i^j + E_r^s\\ \omega_{(p,A)}\left(\frac{\partial}{\partial x_k} + (E_r^s)^*\right) &= \sum_{i,j=1}^{n}\Gamma^i_{kj}\left(Ad_{A^{-1}}\cdot E_i^j\right) + E_r^s\end{aligned}$$

where $(E_i^j)^*_{(p,B)} = \frac{d}{dt}\big|_{t=0}(p, B\exp(tE_i^j))$. The second row comes from $(R_A)_*\left(\frac{\partial}{\partial x_k}\right) = \left(\frac{\partial}{\partial x_k}\right)$ for all $k\in\{1,\ \dots\ ,n\}$.

Theorem 1.1.27 ([KN63, p. 145]). *Let* $\chi = (x^1,\ \dots\ ,x^n)$ *be a coordinate system of* M *and* Γ *be a linear connection. Then, the Christoffel symbols* Γ^i_{kj} *of* Γ *with respect to* χ *are given by,*

$$\nabla_{\frac{\partial}{\partial x_k}}\frac{\partial}{\partial x_j} = \sum_{i=1}^{n}\Gamma^i_{kj}\frac{\partial}{\partial x_i}.$$

1.2 Reductive Homogeneous Manifolds

Dedicated to Oldřich Kowalski.

Definition 1.2.1. A *homogeneous manifold* is a quotient G/H of a Lie group G by a closed subgroup H, endowed with the unique differentiable structure making $\pi\colon G \longrightarrow G/H$ a submersion.

The Lie group G acts on the left on the differentiable manifold $M = G/H$ as,

$$g_1 \cdot [g_2]_H = [g_1 \cdot g_2]_H,$$

for all $g_1 \in G$ and $[g_2]_H \in G/H$. The action above is $\mathcal{C}^\infty$ and transitive.

Conversely, if M is a differentiable manifold and G is a Lie group acting on the left on M in such a way that the action is $\mathcal{C}^\infty$ and transitive, then we can consider the *isotropy group* at any point $p \in M$,

$$H_p = \Big\{ h \in G \colon h \cdot p = p \Big\},$$

which is a closed subgroup of G, and the map

$$\begin{aligned} G/H_p &\longrightarrow M \\ [g]_H &\longmapsto g \cdot p \end{aligned}$$

defines a diffeomorphism between G/H_p and M.

Therefore, Def. 1.2.1 is equivalent to the following Definition:

Definition 1.2.2. A *homogeneous manifold* M is a differentiable manifold such that there exists a Lie group G acting on the left on M and the action is $\mathcal{C}^\infty$ and transitively.

Remark 1.2.3. The uniqueness of this differentiable structure in Def. 1.2.1 comes from [Che46, pp. 109-111]. Moreover, he also proves that the action is not just $\mathcal{C}^\infty$, it is also real analytic.

Most of the classical manifolds that have been studied are homogeneous, for example,

- Lie groups: $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{T}^n$, ...
- Spheres: $\mathbb{S}^n = SO(n+1)/SO(n)$.
- Hyperbolic spaces: $\mathbb{H}^n = SO(n,1)/SO(n)$.

- Projective spaces: $\mathbb{CP}^n = U(n+1)/U(n)\times U(1)$.
- Real Grassmannians: $SO(p+q)/SO(p)\times SO(q)$.
- Complex Grassmannians: $U(p+q)/U(p)\times U(q)$.

We denote by $L_g(p)$ or $g\cdot p$ the action of an element $g\in G$ on $p\in M$. We say that the action of G in M is *effective* if the normal Lie subgroup of G,

$$N=\left\{g\in G:\, L_g = Id_M\right\}$$

is the identity. We can certainly assume that G acts effectively on M, if not, we shall replace G by G/N. From now on, we consider G a Lie group acting transitively and effectively on M.

Definition 1.2.4. Let M be a manifold and $P_1,\dots,P_r$, $r\in\mathbb{N}$, be a finite set of tensor fields on M. We say that $(M,P_1,\dots,P_r)$ is homogeneous if there exists a Lie group G acting $\mathcal{C}^\infty$ and transitively on M and preserving the tensor fields $P_1,\dots,P_r$.

Note that Def. 1.2.4 is a sharpened version of *pseudo-Riemannian homogeneous manifolds* when $P_1=g$ is a metric tensor. Moreover, if we consider a pseudo-Hermitian manifold (M,g,J) (see Sec. 2.1), the additional structure is defined by two tensors, $P_1=g$ the metric and $P_2=J$ the complex tensor. Thus, Def. 1.2.4 also generalized the classical definition of *pseudo-Hermitian homogeneous manifold*.

Remark 1.2.5. In this thesis, for the sake of brevity and simplicity, we consider M with one tensor $K=(P_1,\ \dots\ ,P_r)$ instead of a finite set of tensors, the following results being analogous for $(P_1,\ \dots\ ,P_r)$.

The advantage of using this philosophy of homogeneous manifolds leaving a tensor field invariant is the fact that we can use this definition indistinctly for metric and non-metric manifolds. For example, if $K=\omega$ is a non-degenerate two form, then (M,ω) would be an *almost symplectic homogeneous manifolds*. Moreover, when ω is closed, it is a *symplectic homogeneous manifold*, see Sec. 5.1. In general, most examples of differentiable manifolds are equipped with additional tensors, that is, pseudo-Riemannian, symplectic, complex, pseudo-Kähler, para-Kähler, contact metric, ...

In a more general setting,

Definition 1.2.6. A manifold (M,K) is *locally homogeneous* if there is a Lie pseudo-group acting $\mathcal{C}^\infty$ and transitively on M and preserving the tensor field K.

Recall that, a *pseudo-group* (see [Spi92]) is a collection of local diffeomorphisms $\mathcal{G}$ acting on M, $\varphi\colon U_\varphi\longrightarrow\varphi(U_\varphi)\subset M$ such that:

- Identity: $Id_M \in \mathcal{G}$.
- Inverse: If $\varphi \in \mathcal{G}$, then $\varphi^{-1} \in \mathcal{G}$.
- Restriction: If $\varphi \in \mathcal{G}$, $\varphi\colon U \longrightarrow M$ and $V \subset U$, then $\varphi|_V \in \mathcal{G}$.
- Continuation: If $\mathrm{dom}(\varphi) = \bigcup U_k$ and $\varphi|_{U_k} \in \mathcal{G}$, then $\varphi \in \mathcal{G}$.
- Composition: If $\varphi, \psi \in \mathcal{G}$ and $\mathrm{im}(\varphi) \subset \mathrm{dom}(\psi)$, then $\psi \circ \varphi \in \mathcal{G}$.

Let us consider that if $\mathcal{G}$ is a pseudo-group and there exists a non-zero tensor field K such that every local diffeomorphism $f \in \mathcal{G}$ satisfies,

$$f_* K = K,$$

that is, the elements of the pseudo-group $\mathcal{G}$ are solutions of a system of PDEs. Thus, $\mathcal{G}$ is a Lie pseudo-group. Otherwise, if $K = \{0\}$, then we only consider the case when $\mathcal{G}$ is a Lie pseudo-group. We refer the reader to [Spi92] or [Acc21] for a formal definition and an exposition of this topic. In particular, a Riemannian manifold is locally homogeneous if the pseudo-group of local isometries acts transitively on it.

A *symmetric space* is a homogeneous manifold G/H with an involutive automorphism $\sigma \neq id_G$ of G such that $\sigma(h) = h$ for all $h \in H$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. We consider $\sigma_{*,e} : \mathfrak{g} \longrightarrow \mathfrak{g}$ the differential map of σ in the neutral element $e \in G$. Since $\sigma_{e,*}$ is involutive, we decompose $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ as the eigenspace for 1 (this is, the Lie algebra $\mathfrak{h}$) and -1, respectively. This is called *the Cartan decomposition* of the symmetric space $(G/H, \sigma)$. In particular, this decomposition satisfies the following equalities:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{1.4}$$

The classical work here is [KN69, Ch. XI]. Symmetric spaces are a special type of homogeneous manifolds.

Definition 1.2.7. Let $M = G/H$ be a homogeneous manifold. We say M is a *reductive homogeneous manifold* if the Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-subspace, that is, $\mathrm{Ad}_h(\mathfrak{m}) \subset \mathfrak{m}$ for all $h \in H$.

The condition $\mathrm{Ad}_h(\mathfrak{m}) \subset \mathfrak{m}$, for all $h \in H$ implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The converse is true if H is connected. This definition shows that every symmetric manifold is reductive homogeneous with the Cartan decomposition.

When reductive homogeneous manifolds are considered, we can identify $\mathfrak{m}$ with any T_pM, for all $p \in M$. The identification is given by the isomorphism,

$$\begin{aligned} \mathfrak{m} &\longrightarrow T_pM \\ X &\longmapsto \frac{d}{dt}\Big|_{t=0} \exp(tX)\cdot p. \end{aligned}$$

We usually relate $\mathfrak{m}$ with the tangent space in the neutral element eH. In general, if we take $X \in \mathfrak{g}$ we can define a vector field

$$X_p^* = \frac{d}{dt}\Big|_{t=0} \exp(tX)\cdot p$$

which is called the *infinitesimal generator of* X. These vector fields satisfy,

$$\begin{aligned} (L_g)_*(X^*) &= (\mathrm{Ad}_g(X))^*, \quad & g \in G, X \in \mathfrak{g}, \\ [X^*,Y^*] &= -[X,Y]^*, & X,Y \in \mathfrak{g}. \end{aligned}$$

Additionally, if we have a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{m}$, then $\{(X_1)_p^*, \ldots, (X_n)_p^*\}$ defines a basis of T_pM for any $p \in M$.

Besides that, every Riemannian homogeneous manifold is reductive [CC19, p. 36] and the construction of the Lie algebra decomposition is very useful to understand why this does not work in general, for example, in pseudo-Riemannian geometry. However, in the proof of [CC19, p. 36], we can realize that most pseudo-Riemannian homogeneous manifolds are reductive. In details, if we have a $M = G/H$ a pseudo-Riemannian homogeneous manifold with signature (r,s), then we can consider the non-degenerate symmetric bilinear form on $\mathfrak{g}$ defined by

$$\phi(X,Y) = -B(A_{X^*}, A_{Y^*}),$$

where B is the killing form of $\mathfrak{so}(r,s)$ and $A_{X^*} = \mathcal{L}_{X^*} - \nabla_{X^*}$ is called the *Kostant operator* with ∇ the Levi-Civita connection. If ϕ is non-degenerate on $\mathfrak{h}$, then we can consider

$$\mathfrak{m} = \mathfrak{h}^\perp = \Big\{X \in \mathfrak{g} : \phi(X,\mathfrak{h}) = 0\Big\}$$

and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is a reductive decomposition. As B is non-degenerate, then ϕ would be degenerate on $\mathfrak{h}$ if and only if the subspace $\{A_{X^*} : X \in \mathfrak{h}\} \subset \mathfrak{so}(r,s)$ is degenerate.

We consider the Grassmannian $\mathrm{Gr}(h, \mathfrak{so}(r,s))$ where $\dim\{A_{X^*} : X \in \mathfrak{h}\} = l$; then the submanifold defined by all degenerate subspaces of $\mathfrak{so}(r,s)$ has dimension less than the Grassmannian. Therefore, we shall think that there are much more reductive than non-reductive homogeneous manifolds. Nevertheless, this conclusion is not clear at all, because the Grassmannian does not coincide with the class of homogeneous manifolds. For a deeper discussion of non-reductive homogeneous manifolds, we refer the reader to [CC19, Ch. 7].

Example 1.2.8. Finally, we show a concrete non-metric example. In particular, it is related to the key word "*special symplectic holonomy*". We follow [Sch94, Example 2.1]. Let

$$G = \left\{ \begin{pmatrix} & A & & x \\ & & & y \\ & 0 & 0 & 1 \end{pmatrix} : A \in \mathrm{Sp}(2,\mathbb{R}), x, y \in \mathbb{R} \right\}$$

be the *group of unimodular motions of* $\mathbb{R}^2$. Its Lie algebra $\mathfrak{g}$ is generated by,

$$\begin{aligned} Z_0 &= \frac{9}{4}E_{23}, & Z_1 &= -\frac{3}{2}(E_{13} - E_{21}), \\ Z_2 &= -3(E_{11} - E_{22}), & Z_3 &= -18E_{12}, \end{aligned}$$

and

$$Y = 2E_{13} + E_{21}$$

where E_{ij} is the 3×3-matrix with 1 in the position (i,j) and all other entries are zero. Now, we compute its Lie brackets,

$$\begin{aligned} &[Z_0, Z_1] = 0, && [Z_0, Z_2] = -3Z_0, && [Z_0, Z_3] = -9Z_1 + \frac{27}{2}Y, && [Z_0,\, Y] = 0, \\ &[Z_1, Z_2] = -3Z_1 - \frac{9}{2}Y, && [Z_1, Z_3] = -9Z_2, && [Z_1,\, Y] = 2Z_0, \\ &[Z_2, Z_3] = -6Z_3, && [Z_2,\, Y] = 4Z_1, \\ &[Z_3,\, Y] = 6Z_2. \end{aligned}$$

We can decompose $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h} = \mathrm{span}\{Y\}$ and $\mathfrak{m} = \mathrm{span}\{Z_0, Z_1, Z_2, Z_3\}$. If we consider $H = \{\exp(tY) : t \in \mathbb{R}\}$, then H is a connected and closed subgroup of G. Therefore, as a consequence of $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, the homogeneous manifold $M = G/H$ is reductive with Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

Furthermore, we can endow this manifold M with a symplectic tensor, see Sec. 5.1. We consider the almost symplectic tensor element of $\mathfrak{m}$,

$$\omega_a = 3a\, z_0 \wedge z_3 - a\, z_1 \wedge z_2$$

for $a \in \mathbb{R} - \{0\}$ and $\{z_0, z_1, z_2, z_3\}$ is a dual basis of $\{Z_0, Z_1, Z_2, Z_3\}$. As a consequence of $[\mathfrak{h}, \omega] = 0$ and the identification above $\mathfrak{m} = T_{eH}M$, then we can define a left invariant almost symplectic form on M,

$$(\omega_a)_p(X,Y) = (\omega_a)_{eH}\left((L_g)_*X,(L_g)_*Y\right),$$

that is, M is a homogeneous almost symplectic manifold. This example follows in Ex. 1.2.17.

1.2.1 The canonical connection

Definition 1.2.9. A linear connection ∇ on $M = G/H$ is G-invariant if and only if L_g is an affine map for any $g \in G$, or equivalently, its natural lift $\tilde{L}_g$ maps $P(u_0)$ to $P(\tilde{L}_g(u_0))$ for any $u_0 \in \mathcal{L}(M)$.

The aim of this section is to introduce a few essential aspects of the theory of invariant connections. In particular, the definition and properties of the canonical connection. In the first instance, we do a preparation that does not involve invariant connections.

Note that $(G \longrightarrow G/H,H)$ is a principal bundle. We claim that it is a reduction of the frame bundle $\mathcal{L}(M)$ for $M = G/H$. We consider the Lie group homomorphism,

$$\begin{aligned} H &\longrightarrow \mathrm{Aut}(T_oM) \\ h &\longmapsto (L_h)_{*,o} \end{aligned}$$

where $o = [e]_H$ ($[e]_H$ is the projection of the neutral element of G onto M) and $\mathrm{Aut}(T_oM)$ is the group of linear automorphism of T_oM. This group homomorphism is called the *linear isotropy representation* of H on M. Note that, if the isotropy representation is faithful, or equivalently, if the action of G on $\mathcal{L}(M)$ is free, then the action of G on M is effective. If G is a group that acts as isometries, or at least preserves a connection, then the linear isotropy representation is necessarily faithful. This scenario is precisely the case we are considering in our study. Therefore, from now on, we assume that the isotropy representation is faithful.

Let $u_0 \in \mathcal{L}(M)$ be a reference with $\pi(u_0) = o$. Then, we can identify $\mathbb{R}^n$ with T_oM via the linear isomorphism given by u_0, and we can view the linear isotropy representation as the homomorphism,

$$\begin{aligned} \varphi\colon H &\longrightarrow \mathrm{GL}(n) \\ h &\longmapsto (u_0)^{-1} \circ (L_h)_{*,o} \circ u_0. \end{aligned}$$

We denote by λ the corresponding homomorphism of Lie algebras $\lambda\colon \mathfrak{h} \longrightarrow \mathfrak{gl}(n)$ such that $\lambda = \varphi_{*,e}$. In particular, as the isotropy representation is faithful, the map $\varphi\colon H \longrightarrow \varphi(H)$ is a Lie group isomorphism. Moreover, the Lie group G is included in $\mathcal{L}(M)$ via the differentiable map $g \longmapsto \tilde{L}_g(u_0)$ and, analogously, this map is injective because the action is free. Note that this identification depends on the choice of u_0.

This construction shows that any linear invariant connection reduces to $(G \longrightarrow G/H, H)$, and to clarify the converse question for reductive homogeneous manifolds, we introduce the following theorem.

Theorem 1.2.10 ([Nom54, p. 43, Thm. 8.1]). *Let $M = G/H$ be a reductive homogeneous manifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then, there is a bijective correspondence between invariant connections on $(\mathcal{L}(M) \longrightarrow M, \mathrm{GL}(n))$ and linear maps $\Lambda_{\mathfrak{m}} \colon \mathfrak{m} \longrightarrow \mathfrak{gl}(n)$ such that,*

$$\Lambda_{\mathfrak{m}}(\mathrm{Ad}_h(X)) = \mathrm{Ad}_{\lambda(h)}(\Lambda_{\mathfrak{m}}(X)), \quad X \in \mathfrak{m}, h \in H.$$

The correspondence can be read from

$$\omega_{u_0}(\widetilde{X_o^*}) = \begin{cases} \Lambda_{\mathfrak{m}}(X), & X \in \mathfrak{m} \\ \lambda(X), & X \in \mathfrak{h} \end{cases}$$

where ω is the connection form on $(\mathcal{L}(M) \longrightarrow M, \mathrm{GL}(n))$.

Definition 1.2.11. The invariant connection associated with the linear map $\Lambda_{\mathfrak{m}} = 0$ is called the *canonical connection* associated with the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

This connection is the only one that satisfies some classical properties. The best general reference here is [KN69, Ch. X], and complementary to this reference see [Kow80, Ch. I] (but avoid the section "Algebraic characterization" because the main theorem is wrong).

Proposition 1.2.12 ([KN69, p. 192, Prop. 2.4]). *Let $f_t(p) = \exp(tX) \cdot p$ be the flow of the infinitesimal generator $X \in \mathfrak{m}$ and let $\tilde{f}_t$ be the natural lift. Then, the canonical connection on $M = G/H$ associated with the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is the unique invariant connection such that the orbit $\tilde{f}_t(u_0)$ is horizontal.*

As a consequence, for any fixed $p \in M$, an integral curve $\alpha_t = \exp(tX) \cdot p$ satisfies that parallel transport along α_t is given by the linear map $(L_{\exp(tX)})_*$ on p and α_t is a complete geodesic.

Let now (M, ∇) be a connected manifold with a linear connection ∇ and let $P(u_0) \subset \mathcal{L}(M)$ be the holonomy bundle, for a fixed $u_0 \in \mathcal{L}(M)$. We define the *Lie group of transvections* $\mathrm{Tr}(M, \nabla)$ as the group of affine diffeomorphisms f of M such that its natural lifts (see Eq. (1.1)) $\tilde{f}$ leave every holonomy bundle invariant $P(u) \subset \mathcal{L}(M)$, for all $u \in \mathcal{L}(M)$. Note that, in particular, the group of transvections is a normal connected subgroup of the group of affine diffeomorphisms (see [Kow80, p. 36]).

An affine transformation f of (M, ∇) belongs to $\mathrm{Tr}(M, \nabla)$ if and only if for every point $p \in M$ there is a piecewise differential curve α joining p with $f(p)$ such that the tangent map $f_* \colon T_pM \longrightarrow T_{f(p)}M$ coincides with the parallel transport along α.

Theorem 1.2.13 ([Kow80, Thm. I.25]). *Let $(M, \tilde{\nabla})$ be a connected manifold with a linear connection. Then the following two conditions are equivalent:*

- *The transvection group $\mathrm{Tr}(M, \tilde{\nabla})$ acts transitively on each holonomy bundle $P(u) \subset \mathcal{L}(M)$.*
- *M can be expressed as a reductive homogeneous space G/H with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where G is effective on M, and $\tilde{\nabla}$ is the canonical connection.*

From this theorem, we deduce: If M is a reductive homogeneous space, then, for every piecewise differential curve α joining p with q there exists a global transvection f such that $f(p) = q$ and the tangent map $f_*\colon T_pM \longrightarrow T_qM$ coincides with the parallel transport of the canonical connection along some α.

In Sec. 1.1, we showed that parallel transport is intimately related to covariant derivatives. Therefore, we exhibit some necessary conditions in terms of covariant derivatives for a connection to be the canonical connection.

Proposition 1.2.14 ([CC19, p. 39]). *Let K be a tensor field on M such that K is invariant under the action of G. Then, the tensor field K is parallel with respect to the canonical connection, that is,*

$$\tilde{\nabla} K = 0.$$

In particular, the curvature and torsion of the canonical connection are very rigid.

Theorem 1.2.15 ([KN69, pp. 190-193]). *Let $M = G/H$ be a reductive homogeneous manifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and let $\tilde{\nabla}$ be the canonical connection. Then, the torsion and curvature tensors of $\tilde{\nabla}$ at $o = [e]_H$ are given by,*

$$\tilde{T}_o(X,Y) = -[X,Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m},$$
$$\tilde{R}_o(X,Y)Z = -[[X,Y]_{\mathfrak{h}}, Z], \quad X, Y, Z \in \mathfrak{m}.$$

Furthermore, the torsion and curvature tensors are invariant, that is,

$$\tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}\tilde{R} = 0. \tag{1.5}$$

In the following theorem, we show the converse question: When is a connection satisfying the necessary conditions, that is, Equations (1.5), a canonical connection?

Theorem 1.2.16 ([KN69, Thm. 2.8]). *Let M be a connected and simply-connected manifold. Then, the following assertions are equivalent:*

- *The manifold M is reductive homogeneous.*
- *The manifold M admits a complete linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0,$$

where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion tensor fields of the connection $\tilde{\nabla}$.

In order to compute the covariant derivative of the canonical connection, we can use the following equation at the point $o = [e]_H$ (cf. [KN69, p. 188-192] or [TV83, p. 20]),

$$\tilde{\nabla}_X Y = -[X,Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m} \tag{1.6}$$

or equivalently,

$$\tilde{\nabla}_{X_o^*} Y_o^* = [X_o^*, Y_o^*].$$

Example 1.2.17. We continue with the Ex. 1.2.8. We now compute the curvature, torsion and covariant derivative of the canonical connection. First, we have that,

$$\begin{array}{llll}
\tilde{\nabla}_{Z_0^*} Z_0^* = 0, & \tilde{\nabla}_{Z_0^*} Z_1^* = 0, & \tilde{\nabla}_{Z_0^*} Z_2^* = 3Z_0^*, & \tilde{\nabla}_{Z_0^*} Z_3^* = 9Z_1^*, \\
\tilde{\nabla}_{Z_1^*} Z_0^* = 0, & \tilde{\nabla}_{Z_1^*} Z_1^* = 0, & \tilde{\nabla}_{Z_1^*} Z_2^* = 3Z_0^*, & \tilde{\nabla}_{Z_1^*} Z_3^* = 9Z_1^*, \\
\tilde{\nabla}_{Z_2^*} Z_0^* = -3Z_0^*, & \tilde{\nabla}_{Z_2^*} Z_1^* = -3Z_0^*, & \tilde{\nabla}_{Z_2^*} Z_2^* = 0, & \tilde{\nabla}_{Z_2^*} Z_3^* = 6Z_3^*, \\
\tilde{\nabla}_{Z_3^*} Z_0^* = -9Z_1^*, & \tilde{\nabla}_{Z_3^*} Z_1^* = -9Z_1^*, & \tilde{\nabla}_{Z_3^*} Z_2^* = -6Z_3^*, & \tilde{\nabla}_{Z_3^*} Z_3^* = 0.
\end{array}$$

Secondly, using Thm. 1.2.15, we compute the torsion and curvature of $\tilde{\nabla}$ in $\mathfrak{m}$,

$$\tilde{T} = 3z_0 \wedge z_2 \otimes Z_0 + 9z_0 \wedge z_3 \otimes Z_1 + 3z_1 \wedge z_2 \otimes Z_0 + 9z_1 \wedge z_3 \otimes Z_1 + 6z_2 \wedge z_3 \otimes Z_3$$

and

$$\tilde{R}_{XY} = \frac{-9}{2a}\omega(X,Y) \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{End}(\mathfrak{m}).$$

1.3 On the history of Ambrose-Singer Theorems

Dedicated to Franco Tricerri and his family.

This section focuses on the hierarchy of homogeneous or locally homogeneous Riemannian manifolds. This is, Riemannian manifolds with a transitive and invariant action given by a Lie group or a Lie pseudo-group. In this setting, there are "*three kings or queens*": the Euclidean space, the sphere, and the real hyperbolic space. These are the only three Riemannian complete, connected, and simply connected space forms, which means they have constant sectional curvature.

The Euclidean space is the archetype of a differentiable manifold with constant sectional curvature equal to zero. This means that its geometry aligns perfectly with Euclid's postulates. The Sphere is an example with positive sectional curvature and describes elliptic geometry. The Hyperbolic space provides an example of a Riemannian manifold with negative sectional curvature.

These three models have many symmetries, that is, the group of isometric motions is very large. In particular, they are Riemannian symmetric spaces. This introduces the second class of hierarchy, symmetric spaces, that is "*the royal family*". The study of symmetric manifolds has led to many deep and far-reaching results in geometry and topology, including the classification of compact symmetric spaces, the existence of minimal immersions into spheres, and the study of harmonic maps between symmetric spaces. Moreover, symmetric manifolds have significant applications in physics, including soliton equations and Relativity theory.

As previously highlighted, Riemannian symmetric manifolds fall under the category of homogeneous manifolds. This introduces the two last classes in the hierarchy, which we refer to as "*the population*": homogeneous and locally homogeneous Riemannian manifolds. Studying the population allows us to draw conclusions about the entire "*society*" of homogeneous manifolds. Furthermore, we can investigate a particular homogeneous manifold, showcasing the main strength of the Tricerri-Vanhecke research program [TV83], as it provides tools to differentiate homogeneous manifolds solely by examining their transitive actions.

A chronological overview of the Ambrose-Singer Theorem is provided in this section (see Fig. 1.1). It is a fundamental tool to connect geometry, algebra and analysis. In particular, we explore deeper into the philosophy of Ambrose-Singer's Theorem and the connection between these three fields.

As we mentioned above, symmetric spaces are a particular case of homogeneous manifolds with the Cartan decomposition. In this case, the canonical connection coincides with the

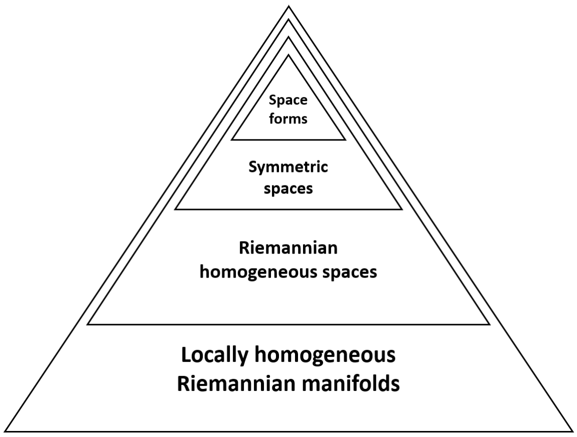

Fig. 1.1 Hierarchy of Riemannian transitive actions

Levi-Civita connection in the Riemannian case. In 1926, Élie Cartan characterized Riemannian symmetric spaces.

Theorem 1.3.1 ([Car29]). *Let (M,g) be a connected and simply-connected complete Riemannian manifold. Then, M is symmetric if and only if $\nabla R = 0$, where R is the curvature tensor of the Levi-Civita connection ∇.*

This theorem provides the following mathematical tools: every symmetric space, whose definition is purely geometric, has a decomposition in terms of Lie algebras called the Cartan decomposition. Furthermore, the covariant derivative of the Levi-Civita connection makes the curvature of this connection parallel.

Following the philosophy of these theorems, the key ideas are shown in the global results where the theorem assumes conditions such as completeness and simply connectedness. However, when we drop these conditions, these theorems remain true at the price that the differentiable manifolds are no longer homogeneous but locally homogeneous. In the case of Cartan's Theorem they would become locally symmetric.

Theorem 1.3.2 ([Car29]). *Let (M,g) be a Riemannian manifold. Then, M is locally symmetric if and only if $\nabla R = 0$, where R is the curvature tensor of the Levi-Civita connection ∇.*

To continue this story, we must look back to 1958. In that year, Ambrose and Singer proved that for complete, connected and simply connected Riemannian homogeneous spaces there was a theorem analogous to Cartan's Theorem.

Theorem 1.3.3 ([AS58, p. 656]). *Let (M,g) be a connected and simply-connected complete Riemannian manifold. Then, the following statements are equivalent:*

1. *The manifold M is Riemannian homogeneous.*

2. *The manifold M admits a linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}g = 0, \tag{1.7}$$

where R is the curvature tensor of the Levi-Civita connection ∇^{LC} and $S = \nabla^{LC} - \tilde{\nabla}$.

Note that, see [TV83, p. 14-16], Eqs. (1.7) are equivalent to:

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}g = 0, \tag{1.8}$$

where $\tilde{R}$ and $\tilde{T}$ are the curvature tensor and torsion tensor of the connection $\tilde{\nabla}$.

Definition 1.3.4. Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. A connection $\tilde{\nabla}$ is a *Ambrose-Singer Riemannian connection* (or *AS-connection*, for short) if it satisfies equations (1.7) (or equivalently, (1.8)). Moreover, under these conditions the difference tensor $S = \nabla^{LC} - \tilde{\nabla}$ is called the *Riemannian homogeneous structure* and the manifold the *Ambrose-Singer Riemannian manifold.*

Riemannian symmetric spaces are reductive homogeneous spaces G/H with G a group of isometries. Moreover, the Cartan decomposition satisfies $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and (1.4). From Thm. 1.2.15, the canonical connection (or AS-connection) for this reductive decomposition is torsion-free and metric, then it is the Levi-Civita connection. It follows that for Riemannian symmetric spaces with the Cartan decomposition, then the homogeneous structure is $S = 0$.

1.3.1 Locally homogeneous Riemannian manifolds

Based on this philosophy, it is necessary to provide an algebraic characterization of locally homogeneous Riemannian manifolds. We take it step by step. In 1960, the following was published:

Theorem 1.3.5 ([Sin60, p. 692])**.** *If M is a complete, simply connected Riemannian manifold which is infinitesimally homogeneous, then M is a Riemannian homogeneous manifold.*

An *infinitesimally homogeneous manifold* is a Riemannian manifold whose covariant derivatives of the curvature tensor of the Levi-Civita connection, up to a certain order, are the same at each point. This would be known as the Second Singer's Theorem, **but it is not true**. If it would be correct, then every locally homogeneous manifold would be locally isometric to a globally homogeneous manifold. Thirty years later, Kowalski found a counter example to the Second Singer's Theorem, see [Kow90]. In 1992, Tricerri proved:

Theorem 1.3.6 ([Tri92, p. 413, Thm. 2.1]). *A Riemannian manifold (M,g) is locally homogeneous if and only if there exists a linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0, \quad \tilde{\nabla} g = 0,$$

where R is the curvature tensor of the Levi-Civita connection ∇^{LC} and $S = \nabla^{LC} - \tilde{\nabla}$.

In [Tri92], there is an example of a locally homogeneous manifold that cannot be locally isometric to any homogeneous manifold. Here ends the brief story concerning the beginning of the study of transitive actions on Riemannian manifolds. And here begins an ambitious program developed by Tricerri and Vanhecke ([TV83], 1983) that provides interesting and powerful geometric results taking advantage of the interplay between partial differential equations, Algebra and Geometry. For a recent reference giving a panoramic view of most of these geometric results, the reader can go to [CC19].

Definition 1.3.7. Let $(M_1, g_1, \tilde{\nabla}_1)$ and $(M_2, g_2, \tilde{\nabla}_2)$ be two Ambrose-Singer manifolds with its respective S_1 and S_2 homogeneous structures. We say that they are *Ambrose-Singer isomorphic* if there exists an isometry $f\colon M_1 \longrightarrow M_2$ such that, $f_* S_1 = S_2$.

Note that f is an isometry, which means it is affine with respect to the Levi-Civita connection of both manifolds. Consequently, f is an affine map between AS-connections.

It is pertinent to note that manifolds may have different representations as quotients, that is, there are different Lie groups acting transitively on the same manifold. **Are these AS-isomorphisms able to differentiate between actions?** Yes, we show that quality of AS-connections with two examples.

- The real hyperbolic space $\mathbb{R}\mathrm{H}(n)$ has more than one representation as a quotient G/H. For example, the symmetric representation of $\mathbb{R}\mathrm{H}(n)$ is $\mathrm{SO}(n,1)/\mathrm{SO}(n)$, with Cartan decomposition, $\mathfrak{so}(n,1) = \mathfrak{so}(n) + \mathfrak{m}$. Thus, the canonical connection is the Levi-Civita connection. Nevertheless, the Lie group representation of $\mathbb{R}\mathrm{H}(n)$ is AN, where A and N are, respectively, the abelian and nilpotent Lie groups of the Iwasawa decomposition of $\mathrm{SO}(n,1)$, (see [TV83, p. 55] and [CGS09, Sec. 3.1]) with a canonical connection of vectorial type given by $\tilde{\nabla} = \nabla + S$ where ∇ is the Levi-Civita connection and S is the homogeneous structure given by,

$$S_X Y = g(X,Y)\xi - g(Y,\xi)X$$

where ξ is the unit killing vector field generated by the abelian part of AN (see [CGS09]).

- From [MS43], all the representations of $\mathbb{S}^n$ as quotient G/H are the following,

$$\begin{aligned} \mathbb{S}^n &= \mathrm{SO}(n+1)/\mathrm{SO}(n), \quad \mathbb{S}^{2n+1} = \mathrm{U}(n+1)/\mathrm{U}(n), \\ \mathbb{S}^{2n+1} &= \mathrm{SU}(n+1)/\,\mathrm{SU}(n) \\ \mathbb{S}^{4n-1} &= \mathrm{Sp}(n)/\mathrm{Sp}(n-1), \quad \mathbb{S}^{4n-1} = \mathrm{Sp}(n+1)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1), \\ \mathbb{S}^{4n-1} &= \mathrm{Sp}(n)\mathrm{U}(1)/\mathrm{Sp}(n-1)\mathrm{U}(1) \\ \mathbb{S}^6 &= G_2/\,\mathrm{SU}(3), \quad \mathbb{S}^7 = \mathrm{Spin}(7)/\,\mathrm{SU}(3), \quad \mathbb{S}^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7), \end{aligned} \tag{1.9}$$

  and every of these representations defines a different (non-isomorphic) canonical connection, see [AHL23].

In light of these examples, we note $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{R}\mathrm{H}(n) = \mathrm{SO}(n,1)/\mathrm{SO}(n)$ have the same expression $S = 0$ homogeneous structure at every point, but they are not even locally isometric. Therefore, it is necessary to develop a new tool to differentiate locally isometric Riemannian manifolds with the same homogeneous structure.

Let $(M, g, \tilde{\nabla})$ be an Ambrose-Singer Riemannian manifold and p be a point in M. Let $V = T_pM$ be the tangent space in a fixed point $p \in M$, and consider

$$\tilde{R}\colon V \wedge V \longrightarrow \mathrm{End}(V), \quad \tilde{T}\colon V \longrightarrow \mathrm{End}(V), \tag{1.10}$$

the curvature and torsion of $\tilde{\nabla}$ on p. Then, the following equations are satisfied

$$\begin{aligned} &\tilde{T}_X Y + \tilde{T}_Y X = 0, \\ &\tilde{R}_{XY} Z + \tilde{R}_{YX} Z = 0, \\ &\tilde{R}_{XY} \cdot \tilde{T} = \tilde{R}_{XY} \cdot \tilde{R} = 0, \\ &\underset{XYZ}{\mathfrak{S}} \tilde{R}_{XY} Z + \tilde{T}_{\tilde{T}_X Y} Z = 0, \\ &\underset{XYZ}{\mathfrak{S}} \tilde{R}_{\tilde{T}_X Y Z} = 0, \\ &\tilde{R}_{XY} \cdot g = 0 \end{aligned} \tag{1.11}$$

where $\underset{XYZ}{\mathfrak{S}}$ is the cyclic sum, and $\tilde{R}_{XY}$ acts in a natural way in the tensor algebra of V as a derivation. This algebraic structure $(T_pM, \tilde{R}, \tilde{T}, g_p)$ is called a *Riemannian infinitesimal model*. In general, a triple $(V, \tilde{R}, \tilde{T}, g_0)$ is called a *Riemannian infinitesimal model* if V is a finite dimensional vector space with a metric tensor element g_0, and $\tilde{R}$, $\tilde{T}$ are two tensor elements defined by (1.10) and satisfy (1.11). Conversely,

Theorem 1.3.8 ([LT93], Thm. 4.1)**.** *For every Riemannian infinitesimal model $(V, \tilde{R}, \tilde{T}, g_0)$ there exists an Ambrose-Singer Riemannian manifold $(M, g, \tilde{\nabla})$ such that, for a point $p \in M$,* *the metric tensor g at p is g_0, and the curvature and torsion of $\tilde{\nabla}$ at p are $\tilde{R}$ and $\tilde{T}$, respectively. In particular, (M,g) is locally homogeneous.*

In addition, we say that two infinitesimal models $(V,\tilde{R},\tilde{T},g)$ and $(V',\tilde{R}',\tilde{T}',g')$ are *isomorphic* if there exists a linear isomorphism $f\colon V \longrightarrow V'$ such that

$$f\tilde{R} = \tilde{R}', \quad f\tilde{T} = \tilde{T}', \quad fg = g'.$$

Obviously, two infinitesimal models associated with different points in an AS-manifold are isomorphic. Moreover, if two different AS-manifolds are AS-isomorphic, then its respective infinitesimal models are isomorphic.

From every Riemannian infinitesimal model $(V,\tilde{R},\tilde{T},g_0)$, we can construct a transitive Lie algebra using the so-called *Riemannian Nomizu construction*, see [Nom54, p. 62]. Let

$$\mathfrak{g}_0 = V \oplus \mathfrak{h}_0, \tag{1.12}$$

where $\mathfrak{h}_0 = \{A \in \mathfrak{so}(V) : A\cdot\tilde{R} = 0, A\cdot\tilde{T} = 0, A\cdot K = 0\}$, equipped with the Lie bracket

$$\begin{aligned} [A,B] &= AB - BA, && A,B \in \mathfrak{h}_0,\\ [A,X] &= AX, && A \in \mathfrak{h}_0, X \in V,\\ [X,Y] &= -\tilde{T}_X Y + \tilde{R}_{XY}, && X,Y \in V. \end{aligned}$$

Alternatively, we can also consider the so-called *transvection algebra*, see [Kow90]. Let

$$\mathfrak{g}_0' = V \oplus \mathfrak{hol} \tag{1.13}$$

where $\mathfrak{hol}$ is the Lie algebra of endomorphisms generated by $\tilde{R}_{XY}$ with $X, Y \in V$, equipped with brackets as above. In particular, this Lie algebra coincides with the holonomy algebra of the connection $\tilde{\nabla}$.

Summarizing, if M is not simply-connected or complete, the existence of $\tilde{\nabla}$ is still extremely useful, as it characterizes locally homogeneous manifolds, a kind of spaces that are more than a mere local version of global spaces.

Example 1.3.9. We consider the Lie algebra representation of $\mathfrak{spin}(n+2)$ given by

$$\mathfrak{spin}(n+2) = \left\{ \begin{pmatrix} 0 & v & a \\ -v^t & A & w^t \\ -a & -w & 0 \end{pmatrix} : \begin{array}{c} A \in \mathfrak{spin}(n); \\ v, w \in \mathbb{R}^n; \\ a \in \mathbb{R} \end{array} \right\}.$$

associated with this representation of Lie algebras, we can construct the decomposition,

$$\mathfrak{spin}(n+2) = \mathfrak{spin}(n) + \mathfrak{m},$$

where

$$\mathfrak{h} = \mathfrak{spin}(n) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix} : A \in \mathfrak{spin}(n) \right\}, \quad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & v^t & a \\ -v & 0 & w \\ -a & -w^t & 0 \end{pmatrix} : \begin{array}{c} v, w \in \mathbb{R}^n; \\ a \in \mathbb{R} \end{array} \right\}.$$

If we include the closed Lie subgroup $\mathrm{SO}(n) = \{\mathrm{diag}(0,A,0) : A \in \mathrm{SO}(n)\} \subset \mathrm{SO}(n+2)$, then we can consider the homogeneous manifold $M = \mathrm{SO}(n+2)/\,\mathrm{SO}(n)$ with reductive decomposition given above, this is a consequence of $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $\mathrm{SO}(n)$ is connected. Moreover, if we lift these Lie groups to their universal covers, then we find a natural way to include $\mathrm{Spin}(n)$ as a closed Lie subgroup of $\mathrm{Spin}(n+2)$. And, analogously,

$$M = \mathrm{Spin}(n+2)/\mathrm{Spin}(n)$$

with dimension equal to $2n+1$ and reductive decomposition given above. The next step is to construct the invariant tensors on the manifold and to construct the homogeneous structure. First, we should notice that there are three invariant $\mathfrak{h}$-submodules of $\mathfrak{m}$ whose expressions are given by,

$$\mathfrak{m}_1 = \left\{ X(v) = \begin{pmatrix} 0 & v^t & 0 \\ -v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : v \in \mathbb{R}^n \right\}, \quad \mathfrak{m}_2 = \left\{ Y(w) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & -w^t & 0 \end{pmatrix} : w \in \mathbb{R}^n \right\},$$

and

$$\mathfrak{n}_2 = \mathrm{span}_{\mathbb{R}}(N_2), \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Since $[\mathfrak{h}, \mathfrak{n}_2] = 0$, there is an invariant vector field ξ on M generated by N_2. We define the 1-form $\eta(\cdot) = \frac{g(\xi,\cdot)}{\sqrt{g(\xi,\xi)}}$. Moreover, we define $\phi\colon \mathfrak{m} \longrightarrow \mathfrak{m}$ such that $\phi(Z) = [N_2, Z]$. In particular,

$$\phi(N_2) = 0, \quad \phi(X(v)) = Y(v), \quad \phi(Y(w)) = -X(w).$$

Therefore, $[\mathfrak{h},\phi]=0$, which means that there exists a left invariant $(1,1)$-tensor field ϕ such that $\phi(\xi)=0$ and $\phi^2=-\mathrm{Id}$. Indeed, these two tensors η and ϕ define an invariant almost contact structure on M. Finally, we construct a left-invariant Riemannian metric g. We take a metric tensor in $\mathfrak{m}$ such that the basis $\{N_2, X(e_i), Y(e_i); i=1,..,n\}$ is an orthogonal basis. In particular, we define $g(B,C)=-\frac{1}{2}\mathrm{tr}(BC)$, for B, $C\in\mathfrak{m}_1+\mathfrak{m}_2$ and $g(N_2,N_2)=\lambda$ for certain $\lambda>0$.

As $[A,X(v)]=X(Av)$ and $[A,Y(w)]=Y(Aw)$ for every $A\in\mathfrak{h}$ and v, $w\in\mathbb{R}^n$, therefore, we have $[\mathfrak{h},\mathfrak{g}]=0$ and we can define a left-invariant Riemannian metric g on M. Indeed, (M,g,ϕ,ξ,η) is a $\mathrm{Spin}(n+2)$-invariant almost contact metric structure, see Sec. 3.1.

Now, we compute the torsion and curvature of the AS-connection. For that, we compute the Lie brackets of the elements in $\mathfrak{m}$:

$$
\begin{aligned}
[X(v_1),X(v_2)] &= \mathrm{diag}(0,-v_1v_2^t+v_2v_1^t,0)\in\mathfrak{h},\\
[Y(w_1),Y(w_2)] &= \mathrm{diag}(0,-w_1w_2^t+w_2w_1^t,0)\in\mathfrak{h},\\
[N_2,X(v)] &= \phi(X(v)),\quad [N_2,Y(w)]=\phi(Y(w)),\\
[X(v),Y(w)] &= g(X(v),X(w))N_2=-g(X(v),\phi(Y(w)))N_2.
\end{aligned}
$$

Therefore, the torsion of the canonical connection is given by

$$\tilde{T}_BC=-[B,C]_{\mathfrak{m}}=g(B,\phi C)\xi-\eta(B)\phi C+\eta(C)\phi B$$

and

$$\tilde{T}_{BCD}=g(-[B,C]_{\mathfrak{m}},D)=g(B,\phi C)\eta(D)+g(C,\phi D)\eta(B)+g(D,\phi B)\eta(C)$$

To get the homogeneous structure tensor we use

$$2g(S_BC,D)=g([B,C]_{\mathfrak{m}},D)-g([C,D]_{\mathfrak{m}},B)+g([D,B]_{\mathfrak{m}},C)$$

(see [CGS06, p. 601] for more details of this formula). Then,

$$
\begin{aligned}
2g(S_BC,D) &= -g(B,\phi C)\eta(D)-g(C,\phi D)\eta(B)-g(D,\phi B)\eta(C)\\
&\quad +g(C,\phi D)\eta(B)+g(D,\phi B)\eta(C)+g(B,\phi C)\eta(D)\\
&\quad -g(D,\phi B)\eta(C)-g(B,\phi C)\eta(D)-g(C,\phi D)\eta(B)\\
&= -g(B,\phi C)\eta(D)-g(C,\phi D)\eta(B)-g(D,\phi B)\eta(C)
\end{aligned}
$$

Therefore,

$$2S_{BCD}=-g(B,\phi C)\eta(D)-g(C,\phi D)\eta(B)-g(D,\phi B)\eta(C),$$

or equivalently,

$$2S_B C = -g(B,\phi C)N_2 + \eta(B)\phi C - \eta(C)\phi B.$$

We study the sectional curvature of the Levi-Civita connection of (M,g). Since

$$R_{BC} = \tilde{R}_{BC} + [S_C, S_B] - S_{\tilde{T}_B C},$$

we have, for unit vector fields,

$$K_{BC} = \tilde{K}_{BC} + g([S_C,S_B]B,C) - g\left(S_{\tilde{T}_B C}B, C\right),$$

where K and $\tilde{K}$ are the sectional curvature of ∇ and $\tilde{\nabla}$, respectively. As,

$$\begin{aligned}
g\big(\tilde{R}_{X(v_1)X(v_2)}X(v_1), X(v_2)\big) &= g\big(\big[[X(v_1),X(v_2)],X(v_1)\big],X(v_2)\big)\\
&= g\big(X(v_1),X(v_1)\big)g\big(X(v_2),X(v_2)\big) - g\big(X(v_1),X(v_2)\big)^2\\
g\big(\tilde{R}_{Y(v_1)Y(v_2)}Y(v_1), Y(v_2)\big) &= g\big(\big[[Y(v_1),Y(v_2)],Y(v_1)\big],Y(v_2)\big)\\
&= g\big(Y(v_1),Y(v_1)\big)g\big(Y(v_2),Y(v_2)\big) - g\big(Y(v_1),Y(v_2)\big)^2
\end{aligned}$$

then, the sectional curvature $\widetilde{K}$ of $\tilde{\nabla}$ is

$$\begin{aligned}
\tilde{K}\big(X(v_1),X(v_2)\big) &= 1, \qquad & \tilde{K}\big(Y(w_1),Y(w_2)\big) &= 1,\\
\tilde{K}\big(X(v_1),Y(w_2)\big) &= 0, & \tilde{K}(N_2,X) &= 0.
\end{aligned} \tag{1.14}$$

Because S is totally skew-symmetric and $\tilde{T}_B C = -2S_B C$,

$$g([S_C,S_B]B,C) = g(S_C S_B B,C) - g(S_B S_C B,C) = g(S_C B, S_B C) = -g(S_B C, S_B C),$$

and

$$-g(S_{\tilde{T}_B C}B,C) = g(S_B \tilde{T}_B C, C) = -g(\tilde{T}_B C, S_B C) = 2g(S_B C, S_B C).$$

Therefore, for unit vector fields,

$$K_{BC} = \tilde{K}_{BC} + g(S_B C, S_B C), \tag{1.15}$$

and

$$\begin{aligned}
g(S_B C, S_B C) = \frac{1}{4}\big(&\lambda g(B,\phi C)^2 + \eta(B)^2 g(\phi C,\phi C)\\
&+ \eta(C)^2 g(\phi B,\phi B) - \eta(B)\eta(C)g(\phi B,\phi C)\big).
\end{aligned}$$

This last equation, evaluated at unit vectors of the basis $\mathfrak{m}$, reads

$$\begin{aligned} g\left(S_{X(v_1)}X(v_2), S_{X(v_1)}X(v_2)\right) &= 0, & g\left(S_{Y(w_1)}Y(w_2), S_{Y(w_1)}Y(w_2)\right) &= 0, \\ g\left(S_{X(v_1)}Y(w_2), S_{X(v_1)}Y(w_2)\right) &= \frac{\lambda}{4}, & g\left(S_{N_2}X, S_{N_2}X\right) &= \frac{\lambda}{4}. \end{aligned} \tag{1.16}$$

For $\lambda = 4$ and unit vectors, we combine (1.15), (1.14) and (1.16). Then, we conclude that the sectional curvature $K(X,Y)$ of the Levi-Civita connection is constant and equal to 1. Indeed, in this case, $M = \mathbb{S}^{2n+1}$.

Specifically, when $n = 7$, we have $M = \mathbb{S}^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$. This setting allows us to compare our constructed description, wherein the isotropy representation has three irreducible submodules $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{n}_2$, with the description presented in [DKL22, pp. 452-454]. In the latter, $\mathrm{Spin}(7)$ acts effectively on two irreducible submodules: one isomorphic to $\mathbb{O}$ and the other to $\mathrm{Im}(\mathbb{O})$. Although these two homogeneous descriptions of $\mathbb{S}^{15}$ are not isomorphic as AS-manifolds, they are isometric. Moreover, both descriptions share the same quotient representation, $\mathbb{S}^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$.

Remark 1.3.10. Furthermore, these examples model the connected, complete and simply-connected homogeneous Sasakian manifolds with homogeneous structures of type $\mathcal{C}_6 \oplus \mathcal{CS}_5$ (see Thm. A.2 and Thm. A.3).

For a deeper discussion on how to compute the homogeneous structure see [TV83] or [CC19, Ch. 2]. Here, we refer to some papers where the authors studied homogeneous structures.

- The three different homogeneous structures of compact Lie groups, see [CC19, p. 56-58].
- The Heisenberg group, see [TV83].
- The standard three sphere, see [CC19, p. 54-55].
- The Berger 3-sphere, see [GO05].
- The spheres, in general, see [AHL23].
- The real hyperbolic spaces, see [CGS09] and [CGS13].
- The complex hyperbolic spaces, see [CC22a].

1.3.2 Classifications of homogeneous structures

A first goal of [TV83] is to determine whether two homogeneous structures can be AS-isomorphic. In fact, the classification of the $(1,2)$-tensor S (also known as *homogeneous* *structure tensor*) into $O(n)$-irreducible classes explicitly specifies necessary conditions. We also denote by S the $(0,3)$-tensor field, $S_{XYZ} = g(S_X Y, Z)$. Let $\mathcal{S}$ be the space of homogeneous pseudo-Riemannian structures. It is decomposed into three classes

$$\begin{aligned} \mathcal{S}_1 &= \Big\{ S \in \mathcal{S} : S_{XYZ} = g(X,Y)\theta(Z) - g(X,Z)\theta(Y),\, \theta \in \Gamma(T^*M) \Big\}, \\ \mathcal{S}_2 &= \Big\{ S \in \mathcal{S} : \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0,\, c_{12}(S) = 0 \Big\}, \\ \mathcal{S}_3 &= \Big\{ S \in \mathcal{S} : S_{XYZ} + S_{YXZ} = 0 \Big\}, \end{aligned} \tag{1.17}$$

where $c_{12}(S)(Z) = \sum_{i=1}^{n} S_{e_i e_i Z}$ for any orthonormal basis $\{e_1, \ldots, e_m\}$.

Note that, $\mathcal{S}$ is a finite submodule of the tensor algebra of M. Furthermore, we can equip $\mathcal{S}$ with a scalar product $\langle , \rangle$ given by

$$\langle S, S' \rangle = \sum_{i,j,k=1}^{n} S_{e_i e_j e_k} S_{e_i e_j e_k},$$

so that, the classes are $O(n)$-irreducible and orthogonal submodules of $\mathcal{S}$.

We deduce from this that there are eight classes of Riemannian homogeneous structures, but we focus on only three.

($\{0\}$) Homogeneous structures with $S = 0$ correspond to Riemannian (locally) symmetric manifolds with their respective Cartan decomposition. See examples above or see [KN69, Chapter X].

($\mathcal{S}_1$) This is called the class of *linear type* and the homogeneous structures of this type are referred to as being of linear type. This is because the dimension of the submodule $\mathcal{S}_1$ grows linearly with the dimension of the manifold. There is a complete description of this class as being locally isometric to the real hyperbolic space, see [TV83, p. 55].

($\mathcal{S}_3$) Homogeneous structures of this class are called *naturally reductive*. They are totally skew-symmetric and the homogeneous structure coincides (up to scalar multiplication) with the torsion tensor of the AS-connection. For example, this type of homogeneous structure arises naturally on the spheres and Lie groups, see [TV83] for more details about naturally reductive manifolds.

1.3.3 Generalizations of the Ambrose-Singer theorem

Important extensions of the Ambrose-Singer Theorem have been carried out in the literature. For example, the characterization of (local) homogeneity on pseudo-Riemannian manifolds was developed in [GO92] and the philosophy of Tricerri and Vanhecke was extended to pseudo-Riemannian geometry in [GO97].

Theorem 1.3.11 ([GO92, p. 454-455, Prop. 1 and Prop. 3]). *Let (M,g) be a connected and simply-connected* ***pseudo****-Riemannian manifold. Then, the following statements are equivalent:*

1. *The manifold M is* ***reductive*** *pseudo-Riemannian homogeneous.*
2. *The manifold M admits a linear* ***complete*** *connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0, \quad \tilde{\nabla} g = 0,$$

where R is the curvature tensor of the Levi-Civita connection ∇^{LC} and $S = \nabla^{LC} - \tilde{\nabla}$.

This situation shows a relevant difference with the original Riemannian version since the existence of the metric connection with parallel torsion and curvature characterizes homogeneous spaces of reductive type only. As we know, the Lie algebra of a group acting transitively on reductive spaces can be decomposed into two factors, invariant under the adjoint action of the isotropy subgroup. Since every Riemannian homogeneous manifold is automatically reductive, this particularity only shows up when dealing with metrics with signature.

The second main extension of the homogeneous structure tensors was given when additional geometric structures are considered together with the pseudo-Riemannian metric, see the following Theorem and its respective references.

Theorem 1.3.12 ([Kir80] or [Luj14, p. 30, Thm. 2.2.4]). *Let (M,g) be a connected and simply-connected pseudo-Riemannian manifold* ***with a geometric structure defined by a set of tensor fields*** *$P_1, \dots, P_k$. Then, the following statements are equivalent:*

1. *The manifold $M = G/H$ is* ***reductive*** *pseudo-Riemannian homogeneous such that G preserves $P_1, \dots, P_k$.*
2. *The manifold M admits a linear* ***complete*** *connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0, \quad \tilde{\nabla} g = 0, \quad \tilde{\nabla} P_i = 0, \quad i = 1, \dots, k$$

where R is the curvature tensor of the Levi-Civita connection ∇^{LC} and $S = \nabla^{LC} - \tilde{\nabla}$.

With geometric structure the authors mean a reduction of the orthogonal frame bundle, that is, a G-structure, to a subgroup G of the orthogonal group of the corresponding signature. This reduction is understood to be determined by the existence of a tensor or set of tensors on the manifold characterizing the frames of the corresponding reduction. From that point of view, the group G is the stabilizer of a canonical tensor (or set of canonical tensors) on $\mathbb{R}^n$ by the natural action of $O(p,q)$, $p+q=n$. When this geometric structure is included in the picture, the notion of homogeneous spaces requires the transitive action of an isometry group that also conserves the geometric tensors on M. Important instances of this situation include Kähler, quaternion-Kähler, Sasaki or G_2 spaces among others.

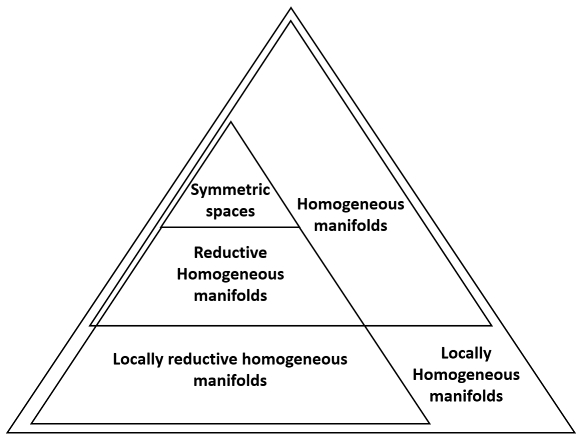

Fig. 1.2 Hierarchy of general transitive actions

Summarising, Riemannian transitive actions have a different hierarchy than transitive actions, in general, where being homogeneous and reductive homogeneous are not the same. Nevertheless, the Tricerri-Vanhecke line of work [TV83] continues in the reductive program even when the manifold is pseudo-Riemannian instead of Riemannian.

Chapter 2

The homogeneous geometries of the complex hyperbolic space

Homogeneous manifolds provide a rich and varied class of spaces that have always deserved the special attention of geometers. In this chapter, we focus on one of these spaces: the complex hyperbolic space $\mathbb{C}\mathrm{H}(n) = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1))$. This manifold plays an important role in different geometric situations (among many others, the reader may look at the following recent works on $\mathbb{C}\mathrm{H}(n)$: [DDS17], [SSS20], and [Won18]) and in particular it is a model for certain questions and classifications. For example, $\mathbb{C}\mathrm{H}(n)$ is the paradigmatic space of the homogeneous structure tensors of linear type in the Ambrose-Singer Theorem for Kähler manifolds since a Kähler manifold with a homogeneous tensor of linear type must be locally holomorphically isometric to the complex hyperbolic space, cf. [GMM00].

A homogeneous geometry of the complex hyperbolic space is understood as a transitive action by isometries of a Lie group G on $\mathbb{C}\mathrm{H}(n)$, that is, a homogeneous description $\mathbb{C}\mathrm{H}(n) = G/H$, together with a canonical connection defined by a reductive decomposition associated with it. Surprisingly, comprehensive lists of all homogeneous descriptions of homogeneous spaces are unknown in many cases. Furthermore, even if all transitive actions on a homogeneous space are known, questions about the geometry of the canonical connections are still unsolved in most cases. In [CGS09], a complete description of the groups G acting transitively on real or complex hyperbolic spaces is provided (so then $\mathbb{R}\mathrm{H}(n)$ or $\mathbb{C}\mathrm{H}(n)$ are G/H for certain subgroups H). With respect to the real case, the classification is completed by the analysis and characterization of the holonomies of all canonical connections in [CGS13]. Apart from that, there are only a few partial results on classical homogeneous manifolds (Berger 3-sphere [GO05], or the 3-dimensional Heisenberg group [TV83, Ch. 7]). With respect to the complex hyperbolic space, the only well known homogeneous geometry is the symmetric one, where one considers the full Lie group of isometries $G = \mathrm{SU}(n,1)$, and the canonical connection coincides with the Levi-Civita connection.

2.1 Pseudo-Kähler manifolds and their homogeneous structures

Let (M,g) be a pseudo-Riemannian manifold with signature (r,s) equipped with an additional $(1,1)$-tensor field J satisfying $J^2 = -\mathrm{Id}$ and

$$g(X,Y) = g(JX,JY) \quad \forall X,Y \in TM.$$

In these conditions, we say that (M,g,J) is an *almost pseudo-Hermitian manifold* and we call *Kähler form* to the 2-form,

$$\omega(X,Y) = g(X,JY).$$

Finally, we say (M,g,J) is a *pseudo-Kähler manifold* if it satisfies,

$$\nabla\omega = 0$$

where ∇ is the Levi-Civita connection. In a broad sense, pseudo-Kähler manifolds are the trivial submodule of the classification in [GH80]. They decompose the space of $(0,3)$-covariant tensor elements with the same symmetries of the covariant derivative of the Kähler form ($\nabla\omega$) in a fixed point $p \in M$, that is,

$$W = \left\{\alpha \in V^* \otimes V^* \otimes V^* : \alpha(x,y,z) = -\alpha(x,z,y) = -\alpha(x,Jy,Jz), \forall x,y,z \in V\right\}.$$

where $V = T_pM$. This space decomposes into four $\mathrm{U}(r,s)$-irreducible and orthogonal submodules as,

$$W = W_1 + W_2 + W_3 + W_4$$

this decomposition gives sixteen classes of almost pseudo-Hermitian manifolds and its explicit expressions are in [GH80, p. 41].

2.1.1 The Ambrose-Singer equations for Kähler manifolds

Let (M,g,J) be a connected, simply-connected and complete Kähler manifold of dimension $2n$ and let S be a Kähler homogeneous structure tensor (see [AG88]), that is, a (1,2)-tensor field on M such that

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}g = 0, \quad \tilde{\nabla}J = 0, \quad \tilde{\nabla}S = 0,$$

where $\tilde{\nabla} = \nabla - S$, ∇ is the Levi-Civita connection, and R its curvature tensor. We fix a point $p \in M$. The holonomy algebra $\mathfrak{hol}$ of $\tilde{\nabla}$ is generated by the endomorphisms $\tilde{R}_{XY}$ of T_pM, for all X, $Y \in T_pM$, where $\tilde{R}$ is the curvature of $\tilde{\nabla}$. Note that these endomorphisms preserve the tensors R, g and J. Furthermore, if we write $\mathfrak{m} = T_pM$ for certain $p \in M$, by the so-called transvection construction (see [Kow80] and (1.13)), the vector space,

$$\tilde{\mathfrak{g}} = \mathfrak{hol} + \mathfrak{m}$$

can be endowed with a Lie bracket defined by

$$[U,V] = UV - VU, \quad [U,X] = U(X), \quad [X,Y] = \tilde{R}_{XY} - \tilde{T}_X Y,$$

for U, $V \in \mathfrak{hol}$ and X, $Y \in \mathfrak{m}$. Since, (M,g,J) is connected, simply-connected and complete, we get a homogeneous description of $M = \tilde{G}/H$, where $\tilde{G}$ and H are obtained by exponentiating $\tilde{\mathfrak{g}}$ and $\mathfrak{hol}$ respectively. Under these conditions, the connection $\tilde{\nabla}$ is the canonical connection associated with the reductive decomposition of $\tilde{\mathfrak{g}} = \mathfrak{hol} + \mathfrak{m}$ (see [KN69, p. 192]). Recall that, for any homogeneous space $M = G/H$ with reductive description $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, the canonical connection at $e = [H]$ is given by ([TV83, p.20]),

$$\tilde{\nabla}_B C = -[B,C]_{\mathfrak{m}} \tag{2.1}$$

for B, $C \in \mathfrak{m}$, where the vector fields of the covariant derivative are regarded as the infinitesimal generators in M induced by elements of $\mathfrak{m}$. The canonical connection has the property that every left-invariant tensor on M is parallel.

We now work from an infinitesimal point of view. Let $V = T_pM$. We interchangeably work with $(1,2)$-tensors and $(0,3)$-tensors given by the isomorphism,

$$(S_p)_{XYZ} = g((S_p)_X Y, Z), \qquad X, Y, Z \in V.$$

For the sake of convenience, S_p is also denoted simply as S. The condition $\tilde{\nabla}_X g = S_X \cdot g = 0$, is equivalent to the skew symmetry of the last two slots of the tensor above. In addition, the condition $\tilde{\nabla}_X J = S_X \cdot J = 0$ is equivalent to the invariance of the two slots with respect to J. We thus define

$$\mathcal{K}(V) = \left\{ S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY},\, S_{XJYJZ} = S_{XYZ} \right\}.$$

This is the space of *Kähler homogeneous structure tensor elements* which is the generalization of the space of Riemannian homogeneous structures $\mathcal{S}(V)$ (see Sec. 1.3.2) for Kähler manifolds. Following the program of Tricerri and Vanhecke for this geometry, the group $\mathrm{U}(n)$ of unitary transformations of $V \simeq \mathbb{R}^{2n}$ acts on the space of tensors $\mathcal{K}(V)$. With respect to this action, $\mathcal{K}(V)$ can be decomposed in orthogonal and irreducible $\mathrm{U}(n)$-submodules (see [AG88, Thm. 2.1] and [BGO11, Thm. 3.5]) as

$$\mathcal{K}(V) = \mathcal{K}_1(V) \oplus \mathcal{K}_2(V) \oplus \mathcal{K}_3(V) \oplus \mathcal{K}_4(V), \tag{2.2}$$

with explicit expressions in Thm. A.1.

The philosophy behind this classification relies on its invariance: if S belongs to one of these class at p, it also belongs to the same class in the classification at any other point $q \in M$. That is (cf. [CC19, Ch. 4]), it is equivalent to study the symmetries of the homogeneous structure S or those of the tensor element S_p.

2.2 The complex hyperbolic space

We refer to [Gol99] for the basics on complex hyperbolic geometry that we outline now. Let $\hat{h}$ be the pseudo-Hermitian product in $\mathbb{C}^{n+1}$ defined by

$$\hat{h}(X,Y) = \overline{Y_1}X_1 + \ \dots \ + \overline{Y_{n-1}}X_{n-1} + \overline{Y_{n+1}}X_n + \overline{Y_n}X_{n+1}.$$

The choice of this form (the so-called second Hermitian form) instead of the canonical one (the first Hermitian from) is compatible with our choice of notation of the algebra $\mathfrak{su}(n,1)$ given below. In any case, both forms are equivalent under a Cayley transformation. From $\hat{h}$ we define the Riemannian metric $\hat{g}(X,Y) = \mathrm{Re}(\hat{h}(X,Y))$ and the Kähler form $\hat{\omega} = \mathrm{Im}(\hat{h}(X,Y))$.

Let $\pi = \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}P^n$ be the canonical projection over the projective space. The complex hyperbolic space is defined as $\mathbb{C}\mathrm{H}(n) = \pi(V_-)$, where $V_- = \{x \in \mathbb{C}^{n+1} : \hat{h}(x,x) < 0\}$. This is the Siegel domain model of $\mathbb{C}\mathrm{H}(n)$. Unfortunately, this definition does not provide a canonical Kähler structure on $\mathbb{C}\mathrm{H}(n)$. For that purpose, given $\mu > 0$, we consider the anti-de Sitter space

$$H^{2n+1}(\mu) = \left\{ x \in \mathbb{C}^{n+1} : \hat{h}(x,x) = -\mu \right\}.$$

Obviously, $H^{2n+1}(\mu)$ is an embedded submanifold of $\mathbb{C}^{n+1}$ of dimension $2n+1$ such that $\pi(H^{2n+1}(\mu)) = \mathbb{C}\mathrm{H}(n)$. The tangent space at $z \in H^{2n+1}(\mu)$ is $T_zH^{2n+1}(\mu) = \{X \in \mathbb{C}^{n+1} : \hat{g}(z,X) = 0\}$ and in particular the vector field $\xi_z = \frac{1}{\sqrt{\mu}} iz$ belongs to $T_zH^{2n+1}(\mu)$ for any $z \in H^{2n+1}(\mu)$. It is easy to check that the projection $\pi\colon H^{2n+1}(\mu) \longrightarrow \mathbb{C}\mathrm{H}(n)$ is an S^1-principal bundle the fibers of which are the integrable submanifolds of ξ. On the other hand, the vector field ξ induces an orthogonal decomposition

$$T_zH^{2n+1}(r) = T'_zH^{2n+1}(r) \oplus \mathbb{R} \cdot \xi_z,$$

with $T_z'H^{2n+1}(\mu) = \{X \in \mathbb{C}^{n+1} : \hat{h}(X,z) = 0\}$. For any $\mu > 0$, we equip $\mathbb{C}\mathrm{H}(n)$ with a Riemannian metric g and Kähler form ω from $\hat{g}$ and $\hat{\omega}$ respectively, by the pointwise isomorphism

$$\nu = \pi_* \colon T_z'H^{2n+1}(\mu) \longrightarrow T_{\pi(z)}\mathbb{C}\mathrm{H}(n), \tag{2.3}$$

for all $z \in H^{2n+1}(\mu)$.

For any $\mu > 0$, the Levi-Civita connection associated with the metric and complex structure induced by the projection $H^{2n+1}(\mu) \longrightarrow \mathbb{C}\mathrm{H}(n)$ has constant sectional holomorphic curvature equal to $-\frac{4}{\mu}$.

2.2.1 The descriptions of $\mathbb{C}\mathrm{H}(n)$ as a homogeneous manifold

The description of $\mathbb{C}\mathrm{H}(n)$ as a symmetric space is given by the quotient

$$\mathbb{C}\mathrm{H}(n) = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1)),$$

where $\mathrm{SU}(n,1)$, the full set of isometries of the complex hyperbolic space, is the set of complex matrices of dimension $n+1$ preserving the form $\mathrm{diag}(\mathrm{Id}_n, -1)$, and determinant $+1$. We regard the group $\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1))$ as the image of the monomorphism

$$\begin{aligned} \mathrm{U}(n) &\longrightarrow \mathrm{SU}(n,1) \\ U &\mapsto \begin{pmatrix} U & 0 \\ 0 & \det(U)^{-1} \end{pmatrix}. \end{aligned} \tag{2.4}$$

For the sake of simplicity of the computations in the rest of the chapter, it is much more convenient to regard $\mathrm{SU}(n,1)$ as the set of complex matrices of dimension $n+1$, preserving the form $\mathrm{diag}(\mathrm{Id}_{n-1}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, and determinant $+1$. With this choice, the Lie algebras of these groups are

$$\mathfrak{su}(n,1) = \left\{ \begin{pmatrix} B & v_1 & v_2 \\ -v_2^* & z & ib \\ -v_1^* & ia & -\bar{z} \end{pmatrix} : \begin{array}{c} z - \bar{z} + \mathrm{tr}(B) = 0; \\ B \in \mathfrak{u}(n-1); \\ v_1, v_2 \in \mathbb{C}^{n-1}; \\ z \in \mathbb{C};\ a, b \in \mathbb{R} \end{array} \right\},$$

$$\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1)) = \left\{ \begin{pmatrix} B & v & v \\ -v^* & i(a+b) & i(a-b) \\ -v^* & i(a-b) & i(a+b) \end{pmatrix} : \begin{array}{c} 2i(a+b) + \mathrm{tr}(B) = 0; \\ B \in \mathfrak{u}(n-1); \\ v \in \mathbb{C}^{n-1};\ a, b \in \mathbb{R} \end{array} \right\}.$$

As usual, the star $*$ stands for the transpose of the complex conjugate. The reductive decomposition

$$\mathfrak{su}(n,1)=\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(1))+\mathfrak{m}$$

of this symmetric description is given by the Cartan decomposition defined by the involution $\theta : \mathfrak{su}(n,1)\longrightarrow\mathfrak{su}(n,1)$, $\theta(A)=\operatorname{diag}(\mathrm{Id}_{n-1},\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))\cdot A\cdot\operatorname{diag}(\mathrm{Id}_{n-1},\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))$. The $(+1)$-eigenspace is $\mathfrak{k}=\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(1))$ whereas $\mathfrak{m}$ is the (-1)-eigenspace is

$$\mathfrak{m}=\left\{\begin{pmatrix}0 & v & -v\\ v^* & a & ib\\ -v^* & -ib & -a\end{pmatrix} : v\in\mathbb{C}^{n-1};\ a,b\in\mathbb{R}\right\}. \tag{2.5}$$

We now give the other homogeneous descriptions $\mathbb{C}\mathrm{H}(n)=G/H$. First, we consider the Iwasawa decomposition $G=KAN$ as well as its infinitesimal version

$$\mathfrak{su}(n,1)=\mathfrak{k}+\mathfrak{a}+\mathfrak{n},$$

where $\mathfrak{k}=\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(1))$ is the compact part, $\mathfrak{a}=\operatorname{span}_{\mathbb{R}}(A_0)$ is the unique maximal $\mathbb{R}$-diagonalizable subalgebra, $A_0=\operatorname{diag}(0,\ \dots\ ,0,1,-1)$, and $\mathfrak{n}=\mathfrak{n}_1+\mathfrak{n}_2$ is the nilpotent part

$$\mathfrak{n}_1=\left\{\begin{pmatrix}0 & 0 & v\\ -v^* & 0 & 0\\ 0 & 0 & 0\end{pmatrix} : v\in\mathbb{C}^{n-1}\right\},\quad \mathfrak{n}_2=\mathbb{R}N_2,\quad N_2=\begin{pmatrix}0&0&0\\0&0&i\\0&0&0\end{pmatrix}.$$

With respect to these last subspaces, they are the eigenspaces $\mathfrak{n}_1=\mathfrak{g}_\lambda$, $\mathfrak{n}_2=\mathfrak{g}_{2\lambda}$, associated with the set of roots $\Sigma=\{\pm\lambda,\pm2\lambda\}$, $\lambda(A_0)=1$.

Based on the result of Witte on cocompact Lie groups [Wit90], one can determine all Lie groups acting transitively on $\mathbb{C}\mathrm{H}(n)$.

Theorem 2.2.1 ([CGS09, pp. 568-569]). *The connected groups of isometries acting transitively on $\mathbb{C}\mathrm{H}(n)$ are the full isometry group $\mathrm{SU}(n,1)$ and the groups $G=F_rN$, where N is the nilpotent factor in the Iwasawa decomposition of $\mathrm{SU}(n,1)$ and F_r is a connected closed subgroup of $\mathrm{S}(\mathrm{U}(n-1)\mathrm{U}(1))\mathbb{R}\subset\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1))\mathbb{R}$ with non trivial projection to $\mathbb{R}$.*

In the following, we repeatedly make use of the next brackets: For $X\in\mathfrak{n}_1$,

$$\begin{aligned}\left[A_0,X\right]&=X, & \left[A_0,N_2\right]&=2N_2,\\ \left[A_0,\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))\right]&=0, & \left[\mathfrak{n}_1,\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))\right]&=\mathfrak{n}_1,\\ \left[\mathfrak{n}_2,\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))\right]&=0, & \left[\mathfrak{n}_1,\mathfrak{n}_2\right]&=0,\\ \left[\mathfrak{n}_2,\mathfrak{n}_2\right]&=0, & \left[\mathfrak{n}_1,\mathfrak{n}_1\right]&=\mathfrak{n}_2,\end{aligned} \tag{2.6}$$

In particular, the last bracket reads

$$\left[\begin{pmatrix} 0 & 0 & v \\ -v^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & w \\ -w^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -v^*w+w^*v \\ 0 & 0 & 0 \end{pmatrix} = +2\omega_0(v,w)N_2$$

where v, $w \in \mathbb{C}^{n-1}$ and ω_0 is the canonical symplectic (Kähler) form in $\mathbb{C}^{n-1}$.

2.2.2 The Kähler structure of the homogeneous descriptions of $\mathbb{C}\mathrm{H}(n)$

Given a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, we identify the reductive complement $\mathfrak{m}$ with $T_{\pi(z)}\mathbb{C}\mathrm{H}(n)$ for some $z \in H^{2n+1}(r)$, and find the concrete expressions of the metric g and the Kähler form ω in $\mathfrak{m}$. Recall that, see (2.3), the Kähler structure on $T_{\pi(z)}\mathbb{C}\mathrm{H}(n)$ is induced by the following isomorphism

$$\nu = \pi_* \colon T_z'H^{2n+1}(\mu) \longrightarrow T_{\pi(z)}\mathbb{C}\mathrm{H}(n),$$

where $T_z'H^{2n+1}(\mu) = \{X \in \mathbb{C}^{n+1} : \hat{h}(X,z) = 0\}$. Therefore, given $X \in \mathfrak{m}$, we first compute

$$\left.\frac{d}{dt}\right|_{t=0} \exp(\mathrm{tX}) \cdot z = X \cdot z \in T_zH^{2n+1}(\mu),$$

and then we orthogonally project to $T_z'H^{2n+1}(\mu)$, that is,

$$\begin{aligned} \mathfrak{m} &\longrightarrow T_z'H^{2n+1}(\mu) \\ X &\longmapsto X\cdot z - \hat{h}(X\cdot z, z). \end{aligned}$$

We choose $z = \left(0, \ \ldots \ ,0, \sqrt{\frac{\mu}{2}} - \sqrt{\frac{\mu}{2}}\right) \in H^{2n+1}(r)$.

For the symmetric description $\mathbb{C}\mathrm{H}(n) = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1))$ and the reductive decompositon $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ given in (2.5), we easily check that

$$X(a,b,v) = \begin{pmatrix} 0 & v & -v \\ v^* & a & ib \\ -v^* & -ib & -a \end{pmatrix} \longmapsto \begin{pmatrix} 2\sqrt{\frac{\mu}{2}}v \\ a\sqrt{\frac{\mu}{2}} - ib\sqrt{\frac{\mu}{2}} \\ a\sqrt{\frac{\mu}{2}} - ib\sqrt{\frac{\mu}{2}} \end{pmatrix} \in T_z'H^{2n+1}(\mu).$$

For the other descriptions, $G = F_rN$, $H = F_r \cap \mathrm{S}(\mathrm{U}(n-1)\mathrm{U}(1))$, the elements of the reductive complements $\mathfrak{m}$ have always non-trivial projection to $\mathfrak{a}+\mathfrak{n}$ and are thus of the type

$$\begin{pmatrix} U & 0 & -2v \\ 2v^* & a+ic & 2ib \\ 0 & 0 & -a+ic \end{pmatrix}; \, v \in \mathbb{C}^{n-1}; \; a, b \in \mathbb{R},$$

for certain $U \in \mathrm{U}(n-1)$. In this case we have

$$X(a,b,v) = \begin{pmatrix} U & 0 & -2v \\ 2v^* & a+ic & 2ib \\ 0 & 0 & -a+ic \end{pmatrix} \longmapsto \begin{pmatrix} 2\sqrt{\frac{\mu}{2}}v \\ a\sqrt{\frac{\mu}{2}} - ib\sqrt{\frac{\mu}{2}} \\ a\sqrt{\frac{\mu}{2}} - ib\sqrt{\frac{\mu}{2}} \end{pmatrix} \in T_z'H^{2n+1}(\mu).$$

Hence, the pull-back of the metric and the symplectic form to $\mathfrak{m}$ under this identification have the same expression for both cases $G = \mathrm{SU}(n,1)$ and $G = F_rN$. In particular, recalling that the metric and the symplectic form in $T_{\pi(Z)}\mathbb{C}H(n)$ are the projection by ν of the standard ones in $T_z'H^{2n+1}(\mu)$, this pull-back forms are

$$g\left(X_1(a_1,b_1,v_1), X_2(a_2,b_2,v_2)\right) = \mu\left(a_1a_2 + b_1b_2 + 2g_0(v_1,v_2)\right) \tag{2.7}$$

and

$$\omega(X_1(a_1,b_1,v_1), X_2(a_2,b_2,v_2)) = \mu(a_1b_2 - a_2b_1 + 2\omega_0(v_1,v_2)), \tag{2.8}$$

where ω_0 and g_0 are the canonical symplectic (Kähler) and Riemannian metric on $\mathbb{C}^{n-1}$ respectively. Obvioulsy, the complex structure tensor J on $\mathfrak{m}$ is characterized by $\omega(X_1,X_2) = g(X_1,JX_2)$.

2.3 The canonical connections on $\mathbb{C}\mathrm{H}(n)$

For the symmetric homogeneous description of $\mathbb{C}H(n)$, the canonical connection is the Levi-Civita connection (the homogeneous structure tensor S vanishes) so that its holonomy is the holonomy of the Riemannian manifold.

In the following, we confine ourselves to the non-symmetric descriptions $\mathbb{C}H(n) = G/H$ where $G = F_rN$ and $H = F_r \cap \mathrm{S}(\mathrm{U}(n-1)\mathrm{U}(1))$. Since (see (2.4)) $\mathrm{S}(\mathrm{U}(n-1)\mathrm{U}(1)) \simeq \mathrm{U}(n-1)$ is compact and F_r is closed, then H is compact and, hence, reductive. Let $\mathfrak{h}$ be the Lie algebra of H and its reductive decomposition,

$$\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{h}_{ss}$$

where $\mathfrak{h}_0$ is abelian and $\mathfrak{h}_{ss}$ is semi-simple.

Let $\mathfrak{f}_r$ be the Lie algebra of $F_r \subset \mathrm{S}(\mathrm{U}(n-1)\mathrm{U}(1))A$. With the restriction of the positive definite inner product $k(E,E') = \mathrm{Re}(\mathrm{tr}(E^*E'))$, E, $E' \in \mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))+\mathfrak{a}$ to $\mathfrak{f}_r$, we decompose

$$\mathfrak{f}_r = \mathfrak{a}_r + \mathfrak{h}$$

where $\mathfrak{a}_r$ is the orthogonal subspace to $\mathfrak{h}$, that is $k(\mathfrak{a}_r,\mathfrak{h}) = 0$. By Thm. 2.2.1, $\mathfrak{a}_r$ projects to $\mathfrak{a}$ and is of dimension 1, so that we can write it as $\mathfrak{a}_r = \mathbb{R}A_r$ with $A_r = A_0 + H_r$ and $H_r \in \mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1)))$. Since $k(A_0,\mathfrak{h}) = 0$ and $k(A_r,\mathfrak{h}) = 0$, hence $k(H_r,\mathfrak{h}) = 0$. Therefore, $H_r = 0$ or $H_r \notin \mathfrak{h}$.

We claim that $[\mathfrak{a}_r,\mathfrak{h}] = 0$. On the one hand, from the adjoint invariance of k, for every H, $H' \in \mathfrak{h}$,

$$k([A_r,H],H') = -k(A_r,[H',H]) = 0$$

and then $k([A_r,\mathfrak{h}],\mathfrak{h}) = 0$ which means that $[A_r,\mathfrak{h}]$ belongs to $\mathfrak{a}_r$. On the other hand, $[A_r,\mathfrak{h}] = [A_0+H_r,\mathfrak{h}] = [H_r,\mathfrak{h}] \subset \mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))$. Hence, $[A_r,\mathfrak{h}]$ belongs to $\mathfrak{a}_r \cap \mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1)) = \{0\}$ because $\mathfrak{a}_r$ projects non-trivially to $\mathfrak{a}$.

Consequently, $\mathfrak{f}_r$ has the following reductive decomposition,

$$\mathfrak{f}_r = (\mathfrak{a}_r + \mathfrak{h}_0) + \mathfrak{h}_{ss}.$$

In the expression

$$\mathfrak{g} = \mathfrak{f}_r + \mathfrak{n} = (\mathfrak{a}_r + \mathfrak{h}_0) + \mathfrak{h}_{ss} + \mathfrak{n}$$

we define

$$\mathfrak{s} = \mathfrak{a} + \mathfrak{n}, \quad \mathfrak{s}_r = \mathfrak{a}_r + \mathfrak{n}.$$

From (2.6), we have that $[\mathfrak{s},\mathfrak{s}] = \mathfrak{n} = [\mathfrak{s}_r,\mathfrak{s}_r]$.

Every canonical connection $\tilde{\nabla}$ in G/H is equivalent to a choice of an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$ complementary to $\mathfrak{h}$ in $\mathfrak{g}$. The subspace $\mathfrak{m}$ can be regarded as a graph of an $\mathfrak{h}$-equivariant map

$$\varphi_r \colon \mathfrak{s}_r \longrightarrow \mathfrak{h}.$$

If we define the $\mathfrak{h}$-equivariant map

$$\chi_r \colon \mathfrak{s} \longrightarrow \mathfrak{s}_r$$

extending the identity in $\mathfrak{n}$ and mapping A_0 to A_r, we can consider

$$\varphi = \varphi_r \circ \chi \colon \mathfrak{s} \longrightarrow \mathfrak{h},$$

and $\mathfrak{m}$ can be regarded as the image of

$$\begin{aligned} \tilde{\cdot} \equiv \chi_r + \varphi \colon \mathfrak{a} + \mathfrak{n} &\longrightarrow \mathfrak{h} \\ X &\longmapsto \tilde{X}. \end{aligned}$$

Lemma 2.3.1. *The Lie algebra $\mathfrak{k}$ of the holonomy group of the canonical connection $\tilde{\nabla}$ associated with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is*

$$\mathfrak{k} = \varphi_r(\mathfrak{n}) = \varphi(\mathfrak{n}).$$

Proof. The holonomy algebra $\mathfrak{hol}$ is generated by

$$[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}.$$

Let $H_0 = \varphi(A_0)$, $H_{r0} = H_r + H_0$. Then

$$\begin{aligned} \tilde{A}_0 &= A_r + H_0 = A_0 + H_{r0}, \\ \tilde{N}_2 &= N_2 + \varphi(N_2), \end{aligned}$$

by the $\mathfrak{h}$-equivariance of φ we have that H_0, $\varphi(N_2) \in \mathfrak{h}_0$.

The subspace $\mathfrak{m}$ is spanned by $\left[\tilde{A}_0, \tilde{X}\right]$, $\left[\tilde{A}_0, \tilde{N}_2\right]$, $\left[\tilde{N}_2, \tilde{X}\right]$ and $\left[\tilde{X}, \tilde{X}'\right]$, for X, $X' \in \mathfrak{n}_1$. We study the projections to $\mathfrak{h}$ of these brackets. During the proof, we repeatedly make use of (2.6).

With respect to the first,

$$\begin{aligned} \left[\tilde{A}_0, \tilde{X}\right] &= \left[A_0, X\right] + \left[A_0, \varphi(X)\right] + \left[H_{r0}, X\right] + \left[H_{r0}, \varphi(X)\right] \\ &= \quad X \quad + \qquad 0 \qquad + \left[H_{r0}, X\right] + \qquad 0, \end{aligned} \tag{2.9}$$

since $[H_r, \mathfrak{h}] = 0$. As $H_{r0} \in \mathfrak{s}(\mathfrak{u}(n-1) + \mathfrak{u}(1))$, then $[\tilde{A}_0, \tilde{X}]$ lies in $\mathfrak{n}_1$. Furthermore, H_{r0} acts on $\mathfrak{n}_1$ as a skew-Hermitian matrix acts in $\mathbb{C}^{n-1}$, and then $\mathrm{ad}_{H_{r0}}$ has no non-zero real roots. Therefore, the function

$$f := \mathrm{Id} + \mathrm{ad}_{H_{r0}} \colon \mathfrak{n}_1 \longrightarrow \mathfrak{n}_1$$

is invertible. Hence $\left\{[\tilde{A}_0, \tilde{X}] : X \in \mathfrak{n}_1\right\} = \mathfrak{n}_1$. On the other hand, as $X + \varphi(X) \in \mathfrak{m}$, for $X \in \mathfrak{n}_1$, then $(X)_{\mathfrak{h}} = -\varphi(X)$. Consequently,

$$\left\{ \left[\tilde{A}_0, \tilde{X}\right]_{\mathfrak{h}} : X \in \mathfrak{n}_1 \right\} = \varphi(\mathfrak{n}_1).$$

The second bracket yields

$$\begin{aligned}\left[\tilde{A}_0,\tilde{N}_2\right] &= \left[A_0,N_2\right] + \left[H_{r0},N_2\right] + \left[A_0,\varphi(N_2)\right] + \left[H_{r0},\varphi(N_2)\right] = \\ &= \quad 2N_2 \quad + \quad 0 \quad + \quad 0 \quad + \quad 0,\end{aligned}$$

which means that $N_2 = \frac{1}{2}\left[\tilde{A}_0,\tilde{N}_2\right]$. Since $N_2 + \varphi(N_2) \in \mathfrak{m}$, then $(N_2)_{\mathfrak{h}} = -\varphi(N_2)$ and consequently,

$$\left[\tilde{A}_0,\mathfrak{n}_2\right]_{\mathfrak{h}} = \varphi(\mathfrak{n}_2).$$

The third bracket is

$$\begin{aligned}\left[\tilde{N}_2,\tilde{X}\right] &= \left[N_2,X\right] + \left[\varphi(N_2),X\right] + \left[N_2,\varphi(X)\right] + \left[\varphi(N_2),\varphi(X)\right] \\ &= \quad 0 \quad + \left[\varphi(N_2),X\right] + \quad 0 \quad + \varphi\big([N_2,\varphi(X)]\big) = \left[\varphi(N_2),X\right].\end{aligned}$$

As $[\varphi(N_2),X] \in \mathfrak{n}_1$, then $[\tilde{N}_2,\tilde{X}]_{\mathfrak{h}} = -\varphi([\varphi(N_2),X]) = -[\varphi(N_2),\varphi(X)] = -\varphi([N_2,\varphi(X)]) = 0$.

Finally, for $X, X' \in \mathfrak{n}_1$, the fourth bracket is

$$[\tilde{X},\tilde{X}'] = [X,X'] + [X,\varphi(X')] + [\varphi(X),X'] + [\varphi(X),\varphi(X')]. \tag{2.10}$$

The first term lies in $\mathfrak{n}_2$, the second and third lie in $\mathfrak{n}_1$ and the fourth lies in $\mathfrak{h}$. Hence, projecting to $\mathfrak{h}$, we have

$$\begin{aligned}[\tilde{X},\tilde{X}']_{\mathfrak{h}} &= -\varphi([X,X']) - \varphi([X,\varphi(X')]) - \varphi([\varphi(X),X']) + [\varphi(X),\varphi(X')] \\ &= -\varphi([X,X']) - [\varphi(X),\varphi(X')].\end{aligned}$$

As $[\mathfrak{n}_1,\mathfrak{n}_1] = \mathfrak{n}_2$, and $\left[\varphi(\mathfrak{n}_1),\varphi(\mathfrak{n}_1)\right] \subset \left[\mathfrak{h},\varphi(\mathfrak{n}_1)\right] = \varphi\big([\mathfrak{h},\mathfrak{n}_1]\big) \subset \varphi(\mathfrak{n}_1)$, then

$$\left\{\left[\tilde{X},\tilde{X}'\right]_{\mathfrak{h}} : X, X' \in \mathfrak{n}_1\right\} \subset \varphi(\mathfrak{n}_1) + \varphi(\mathfrak{n}_2),$$

and the proof is complete. □

From [KN69, Thm. 2.6], we now get the expressions of the curvature and the torsion forms of all canonical connections $\tilde{\nabla}$ in $\mathbb{C}\mathrm{H}(n)$.

Corollary 2.3.2. *Following the notation above, the curvature form $\tilde{R}$ of a canonical connection $\tilde{\nabla}$ of a non-symmetric description $\mathbb{C}\mathrm{H}(n) = G/H$ is given by,*

$$\begin{aligned}&\tilde{R}_{\tilde{A}_0\tilde{X}} = -\varphi(fX), \quad \tilde{R}_{\tilde{A}_0\tilde{N}_2} = -2\varphi(N_2), \quad \tilde{R}_{\tilde{N}_2\tilde{X}} = 0 \\ &\tilde{R}_{\tilde{X}\tilde{Y}} = -2\omega_0(X,Y)\varphi(N_2) - \left[\varphi(X),\varphi(Y)\right]\end{aligned}$$

where X, $Y \in \mathfrak{n}_1$ *and* $f := \mathrm{Id} + \mathrm{ad}_{H_{r0}}$ *as in the proof of Lem. 2.3.1.*

Corollary 2.3.3. *Following the notation above, the torsion form* $\tilde{T}$ *of a canonical connection* $\tilde{\nabla}$ *of a non-symmetric description* $\mathbb{C}\mathrm{H}(n) = G/H$ *is given by,*

$$\tilde{T}_{\tilde{A}_0}\tilde{X} = -\widetilde{fX}, \quad \tilde{T}_{\tilde{A}_0}\tilde{N}_2 = -2\tilde{N}_2, \quad \tilde{T}_{\tilde{N}_2}\tilde{X} = \big[\varphi(N_2), X\big]$$
$$\tilde{T}_{\tilde{X}}\tilde{Y} = -2\omega_0(X,Y)\tilde{N}_2 - \big[\varphi(X), \tilde{Y}\big] - \big[\tilde{X}, \varphi(Y)\big]$$

where X, $Y \in \mathfrak{n}_1$ *and* $f := \mathrm{Id} + \mathrm{ad}_{H_{r0}}$ *as in the proof of Lem. 2.3.1.*

We also have the following result, it is important in the study of homogeneous structures of linear type.

Corollary 2.3.4. *For any canonical connection of a non-symmetric homogeneous description* $\mathbb{C}\mathrm{H}(n) = G/H$*, there exist two non-vanishing and non-collinear parallel vector fields.*

Proof. As $\tilde{R}_{BC}\tilde{A}_0 = 0$ and $\tilde{R}_{BC}\tilde{N}_2 = 0$, for any B, $C \in \mathfrak{m}$, then we have that $\tilde{\nabla}\tilde{A}_0 = 0$ and $\tilde{\nabla}\tilde{N}_2 = 0$. □

2.4 The homogeneous structure tensors of $\mathbb{C}\mathrm{H}(n)$

The goal of this section is to provide a technical but useful expression of the homogeneous structure tensor $S = \nabla - \tilde{\nabla}$ for any reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of a (non-symmetric) homogeneous description $G/H = \mathbb{C}H(n)$.

First note that the decomposition

$$\mathfrak{m} = \tilde{\mathfrak{a}} + \tilde{\mathfrak{n}}_1 + \tilde{\mathfrak{n}}_2,$$

given by the isomorphism $\tilde{\ } \equiv \chi_r + \varphi \colon \mathfrak{a} + \mathfrak{n} \longrightarrow \mathfrak{m}$ is orthogonal with respect to the metric (2.7). Furthermore, if we write

$$B = \alpha_B\tilde{A}_0 + \eta_B\tilde{N}_2 + \tilde{X}_B, \qquad B \in \mathfrak{m},$$

the symplectic structure (2.8) gives that

$$\alpha_{JB} = \frac{1}{2}\eta_B, \quad \eta_{JB} = -2\alpha_B, \quad \tilde{X}_{JB} = J\tilde{X}_B. \tag{2.11}$$

For later convenience we define

$$B' = \alpha_B\varphi(A_0) + \eta_B\varphi(N_2) + \varphi(X_B),$$

$$B_r = \widetilde{[H_r, X_B]},$$

where $H_r = \chi_r(A_0) = \tilde{A}_0 - \varphi(A_0)$. Note that H_r belongs to $\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))$ so that it preserves g and ω.

Lemma 2.4.1. *For any B, C, $D \in \mathfrak{m}$, we have the following formulas,*

$$\begin{aligned}[B,C]_{\mathfrak{m}} = \alpha_B(C + C_r - \eta_C\tilde{N}_2) - \alpha_C(B + B_r - \eta_B\tilde{N}_2)\\ + [B',C] - [C',B] + \frac{4}{\mu}\omega(B,C)\tilde{N}_2\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}g([B,C]_{\mathfrak{m}},D) = \quad & \alpha_B g(C,D) - \alpha_C g(B,D) + \frac{\mu}{4}\eta_D\left(\alpha_C\eta_B - \eta_C\alpha_B\right)\\ & + \alpha_B g(C_r,D) - \alpha_C g(B_r,D) + \eta_D\omega(B,C)\\ & + g([B',C],D) - g([C',B],D).\end{aligned} \tag{2.13}$$

Proof. From (2.9) and (2.10) we get

$$\begin{aligned}[B,C] = \; & \alpha_B\left(2\eta_C N_2 + X_C + [H_r,X_C] + [H_0,X_C]\right)\\ & - \alpha_C\left(2\eta_B N_2 + X_B + [H_r,X_B] + [H_0,X_B]\right)\\ & + [\tilde{X}_B,\tilde{X}_C] + [\varphi(X_B),X_C] + [X_B,\varphi(X_C)] + [\varphi(X_B),\varphi(X_C)]\\ & + \eta_B[\varphi(N_2),X_C] - \eta_C[\varphi(N_2),X_B].\end{aligned}$$

We add $\pm\alpha_B\alpha_C A_0$, $\pm\alpha_B\eta_C N_2$, $\pm\alpha_C\eta_B N_2$, in first and second rows and we project to $\mathfrak{m}$. We get (2.12) from the expression (2.8) that now looks like

$$2(\alpha_B\eta_C - \eta_B\alpha_C)\tilde{N}_2 + \widetilde{[X_B,X_C]} = \frac{4}{\mu}\omega(B,C)\tilde{N}_2.$$

Finally, (2.13) is a direct consequence. □

Theorem 2.4.2. *Following the notation above, the homogeneous tensor S associated with a canonical connection $\tilde{\nabla}$ reads*

$$\begin{aligned}g(S_B C,D) = \alpha_D g(B,C) - \alpha_C g(B,D) + \alpha_{JD} g(B,JC) - \alpha_{JC} g(B,JD)\\ - \alpha_{JB}\omega(\tilde{X}_C,\tilde{X}_D) + g([B',C],D) + \alpha_B g(C_r,D)\end{aligned} \tag{2.14}$$

for any B, C, $D \in \mathfrak{m}$.

Proof. To compute the homogeneous structure tensor we use

$$2g\big(S_B C, D\big) = g\big([B,C]_{\mathfrak{m}}, D\big) - g\big([C,D]_{\mathfrak{m}}, B\big) + g\big([D,B]_{\mathfrak{m}}, C\big)$$

for B, C, $D \in \mathfrak{m}$, derived from (2.1) and [Bes87, p. 183]. Making use of Lem. 2.4.1 and taking into account that for any $U \in \mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))$ and $\hat{B}$, $\hat{C} \in \mathfrak{s}$, then $g([U,\hat{B}],\hat{C}) + g(\hat{B},[U,\hat{C}]) = 0$, we have

$$\begin{aligned} 2g(S_B C, D) &= 2\alpha_D g(B,C) - 2\alpha_C g(B,D) + \frac{\mu}{2}\eta_B(\alpha_C \eta_D - \alpha_D \eta_C) \\ &\quad + \eta_D \omega(B,C) - \eta_B \omega(C,D) + \eta_C \omega(D,B) \\ &\quad + 2g([B',C],D) + 2\alpha_B g(C_r, D), \end{aligned}$$

The proof is complete by (2.8) and (2.11). □

2.5 The holonomy algebras on $\mathbb{C}\mathrm{H}(n)$

Let $\mathfrak{k}$ be the holonomy algebra of a canonical connection on $\mathbb{C}\mathrm{H}(n)$ different to $\mathfrak{su}(n,1)$, that is $\mathfrak{k} = \varphi(\mathfrak{n})$ for certain $\mathfrak{h}$-equivariant morphism

$$\varphi|_{\mathfrak{n}} \colon \mathfrak{a} + \mathfrak{n} \longrightarrow \mathfrak{h}.$$

We decompose $\mathfrak{n} \simeq \mathbb{C}^{n-1} + \mathbb{R}$ in two orthogonal $\mathfrak{h}$-modules,

$$\mathfrak{n} = V_{\mathfrak{k}} + V' \tag{2.15}$$

where $V' \cong \ker \varphi|_{\mathfrak{n}}$ and $V_{\mathfrak{k}} = \{X \in \mathfrak{n} : \ g(X,V') = 0\} \cong \mathfrak{n}/\ker\varphi|_{\mathfrak{n}}$. Since $\mathfrak{k}$ is a subalgebra of $\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))$, it is of compact type and reductive. We can decompose it as

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_{ss},$$

where $\mathfrak{k}_0$ is abelian and $\mathfrak{k}_{ss}$ is semi-simple. This gives a decomposition of $V_{\mathfrak{k}}$ as

$$V_{\mathfrak{k}} = V_0 + V_{ss}.$$

We now consider the complex tensor J associated with the Kähler structure given above.

Lemma 2.5.1. *The holonomy algebra $\mathfrak{k}$ acts trivially on $V_0 + J(V_0) + \mathfrak{n}_2 + \mathfrak{a}$.*

Proof. Since $\mathfrak{k} \subset s(\mathfrak{u}(n-1)+\mathfrak{u}(1))$, from (2.6) we have $[\mathfrak{k},\mathfrak{a}]=0$ and $[\mathfrak{k},\mathfrak{n}_2]=0$. On the other hand, for $N \in V_0$, we have $[\mathfrak{k},N] \subset V_0$ since V_0 is a $\mathfrak{k}$-module. But $\varphi([\mathfrak{k},N]) = [\mathfrak{k},\varphi(N)] \subset [\mathfrak{k},\mathfrak{k}_0]=0$. Then $[\mathfrak{k},N] \in V' \cap V_0 = \{0\}$. Finally, $\mathfrak{k}$ preserves J, and we get the $[\mathfrak{k},JN]=0$. □

Lemma 2.5.2. *The Lie algebra $\mathfrak{k}_0$ acts trivially on $V_{ss}+J(V_{ss})$.*

Proof. We consider $N \in V_{ss}$, then, $[\mathfrak{k}_0,N] \in V_{ss} \cap V' = \{0\}$. □

Lemma 2.5.3. *There exists a $\mathfrak{k}$-module W isomorphic to V_{ss} and contained in V'.*

Proof. Let $\mathfrak{k}_{ss} = \mathfrak{k}_{s_1} + \ldots + \mathfrak{k}_{s_l}$ be the decomposition of $\mathfrak{k}_{ss}$ in sum of simple Lie algebras and let V_{s_i} be the $\mathfrak{k}$-module associated with $\mathfrak{k}_{s_i}$ for any $i \in \{1, \ldots, l\}$.

First we proof that, for any $i \in \{1, \ldots, l\}$ and any non zero $X \in V_{s_i}$, then $JX \notin V_{ss}$. Suppose that $JX \in V_{ss}$. The sum decomposition gives that $\left[\mathfrak{k}_{s_j},X\right] = \delta_{ij}V_i$, for any j, where δ_{ij} is the Kronecker delta. Then, $\left[\mathfrak{k}_{s_j},JX\right] = \delta_{ij}JV_{s_i}$ and, since $JX \in V_{ss}$, we have $JX = (JX)_1 + \ldots + (JX)_l$ with $JX_j \in V_j$. Therefore, we must have $JV_{s_i} = V_{s_i}$. Let X_B and X_C be two vectors in V_{s_i}. On the one hand, as $\varphi\big(\big[\tilde{X}_B',X_C\big] - \big[X_B,\tilde{X}_C'\big]\big) = 0$ and $\ker\big(\varphi|_{V_{ss}}\big) = \{0\}$, we have $\big[\tilde{X}_B',X_C\big] = -\big[\tilde{X}_C',X_B\big]$. On the other, if we consider X_B, X_C, $X_D \in V_{s_i}$,

$$\big(\big[(\tilde{X}_B)',X_C\big],X_D\big) = g\big(\big[(\tilde{X}_B)',JX_C\big],JX_D\big)$$

as $JX_C \in V_{s_i}$,

$$\begin{aligned} g\big(\big[(\tilde{X}_B)',JX_C\big],JX_D\big) &= -g\big(\big[(J\tilde{X}_C)',X_B\big],JX_D\big) = +g\big(\big[(J\tilde{X}_C)',JX_D\big],X_B\big) \\ &= -g\big(\big[(J\tilde{X}_C)',X_D\big],JX_B\big) = +g\big(\big[(\tilde{X}_D)',JX_C\big],JX_B\big) \\ &= +g\big(\big[(\tilde{X}_D)',X_C\big],X_B\big) = -g\big(\big[(\tilde{X}_B)',X_C\big],X_D\big). \end{aligned}$$

Thus, $g\big(\big[(\tilde{X}_B)',X_C\big],X_D\big) = -g\big(\big[(\tilde{X}_B)',X_C\big],X_D\big) = 0$ for every X_B, X_C, $X_D \in V_{s_i}$ and therefore $\big[(\tilde{X}_B)',X_C\big] = 0$. As $\big[\mathfrak{k}_{s_j},X\big] = \delta_{ij}V_i$, $\mathfrak{k}_{s_i}$ acts effectively on V_{s_i} and trivially on V_{s_j} with $j \neq i$ if $\big[\tilde{X}_B',X_C\big] = 0$ for every X_B, $X_C \in V_{s_i}$ then $\mathfrak{k}_{s_i}$ is necessarily zero, and this is a contradiction with the fact that $0 \neq X \in \mathfrak{k}_{s_i}$.

Now, we can say that, for any $X \in V_{s_i}$, $X \neq 0$, the decomposition of $\mathfrak{k}$-modules $V = V_{ss} + V_0 + V'$, gives $JX = (JX)_{ss} + (JX)_0 + (JX)_{V'}$ with $(JX)_0 + (JX)_{V'} \neq 0$. But, as $\mathfrak{k}_{s_i} \subset \mathfrak{k}_{ss}$ acts trivially on V_0 (Lem. 2.5.1), $JV_{s_i} = \big[\mathfrak{k}_{s_i},JX\big] = \big[\mathfrak{k}_{s_i},(JX)_{ss}\big] + \big[\mathfrak{k}_{s_i},(JX)_{V'}\big] \subset V_{ss} + V'$, we get $(JX)_0 = 0$. We thus define the $\mathfrak{k}$-module $V_{s_i}' = \big[\mathfrak{k}_{s_i},(JX)_{V'}\big] \subset V'$, with $\dim(\mathfrak{k}_{s_i}) = \dim(V_{s_i}) \geq \dim(V_{s_i}')$. The map $\psi_i \colon V_{s_i} \longrightarrow V_{s_i}'$, $\psi_i(X) = (JX)_{V'}$ is a $\mathfrak{k}$-module isomorphism, injective, because of $\ker\psi_i = \{0\}$, and surjective, because of dimensions.

Finally, we consider the $\mathfrak{k}$-module morphism $\psi \colon V_{ss} \longrightarrow V'$, $\psi(X) = (JX)_{V'}$. If $X = X_1 + \ldots + X_l$, $X_i \in \mathfrak{k}_{s_i}$, with $\psi(X) = 0$, then $0 = \big[\mathfrak{k}_{s_i},\psi(X)\big] = \psi([\mathfrak{k}_{s_i},X]) = \psi([\mathfrak{k}_{s_i},X_i]) = \psi_i([\mathfrak{k}_{s_i},X_i])$.

Since ψ_i is injective, we have $[\mathfrak{k}_{s_i}, X_i] = 0$ and then $X_i = 0$ because $\mathfrak{k}_{s_i}$ acts effectively on V_{s_i}, for all i. Then ψ is injective and the image $\psi(V_{ss})$ is the W of the statement. $\square$

Theorem 2.5.4. *The holonomy algebras of canonical connection on* $\mathbb{C}\mathrm{H}(n)$*, are* $\mathfrak{su}(n,1)$ *and all reductive Lie algebras of compact type*

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_{ss}$$

with $\mathfrak{k}_0 \cong \mathbb{C}^r \times \mathbb{R}^s$ *abelian and* k_{ss} *semi-simple where* $s \geq 0$*, and* $r \geq 0$*, satisfying any of the following two constraints of dimensions,*

$$3r + 2s + \dim(\mathfrak{k}_{ss}) \leq n - 1,$$

or, $s \geq 1$ *and*

$$3r + 2(s-1) + 1 + \dim(\mathfrak{k}_{ss}) \leq n - 1.$$

Remark 2.5.5. As we exhibit in the proof, the conditions and structure of $\mathfrak{k}$ provided by this theorem are determined by the role played by $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ with respect to the morphism φ of the equality $\mathfrak{k} = \varphi(\mathfrak{n})$.

First, the existence of two different conditions on the dimensions shows whether $\varphi(N_2) \in \mathbb{R}^s$ vanishes or not. Of course, there are many choices of r, s and $\dim \mathfrak{k}_{ss}$ that satisfy both inequalities. In these cases, the two possibilities $\varphi(N_2) = 0$ and $\varphi(N_2) \neq 0$ can be considered.

On the other hand, the splitting of $\mathfrak{k}_0$ into complex and real parts exhibits the action of J. In particular $\mathbb{C}^r$ is the J-invariant part of $(\ker\varphi)^{\perp} \subset \mathfrak{n}_1 \simeq \mathbb{C}^{n-1}$ from the identification $\mathfrak{k} \simeq (\ker\varphi)^{\perp}$.

Proof. Let $\tilde{\nabla}$ be a canonical connection on $\mathbb{C}\mathrm{H}(n)$ (different to the symmetric one) and $\tilde{R}$ its curvature tensor. As $\mathfrak{k}_0 \cong \mathbb{C}^r \times \mathbb{R}^s$ is Abelian, then an effective metric representation of $\mathfrak{k}_0$ is at least of real dimension $4r + 2s$. On the other hand, $\mathfrak{k}_{ss}$ acts effectively on $V_{ss} + W$ where W is defined in Lem. 2.5.3, and $\dim(V_{ss} + W) = 2\dim(\mathfrak{k}_{ss})$. We know that $\mathfrak{k}$ acts effectively on $\mathfrak{a} + \mathfrak{n}$ and the action is trivial on $V_0 + J(V_0) + \mathfrak{n}_2 + \mathfrak{a}$ (Lem. 2.5.1). Since the actions of $\mathfrak{k}_0$ and $\mathfrak{k}_{ss}$ are nonequivalent, we have

$$4r + 2s + 2\dim(\mathfrak{k}_{ss}) \leq \dim(\mathfrak{a} + \mathfrak{n}) - \dim(V_0 + J(V_0) + \mathfrak{n}_2 + \mathfrak{a}).$$

We distinguish two cases, providing the two inequalities of the statement:

- If we have $\tilde{R}_{\tilde{A}_0\tilde{N}_2} = -2\varphi(N_2) = 0$, then $N_2 \notin V_0$ and $N_2 \notin JV_0$ (since $2JN_2 = A_0 \notin V_0 \subset V_{\mathfrak{k}}$). Then

$$\dim(V_0 + J(V_0) + \mathfrak{n}_2 + \mathfrak{a}) = 2\dim(\mathbb{C}^r + \mathbb{R}^s) - \dim(V_0 \cap JV_0) + 2 = 2(2r+s) - 2r + 2$$

so that

$$6r + 4s + 2\dim(\mathfrak{k}_{ss}) \le 2n - 2,$$

and we get the first inequality.

- If we have $-2\varphi(N_2) = \tilde{R}_{\tilde{A}_0\tilde{N}_2} \neq 0$, as $[\mathfrak{n}_2, \mathfrak{s}(\mathfrak{u}(n-1) + \mathfrak{u}(1))] = 0$ (see (2.6)) and $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{u}(n-1) + \mathfrak{u}(1))$, we have that $\varphi(N_2) \in \mathfrak{k}_0$ or $N_2 \in V_0$, and in fact, $N_2 \in \mathbb{R}^s$. Furthermore $A_0 \in JV_0$. Then

$$\dim(V_0 + J(V_0) + \mathfrak{n}_2 + \mathfrak{a}) = 2\dim(\mathbb{C}^r + \mathbb{R}^s) - \dim(V_0 \cap JV_0) = 2(2r+s) - 2r$$

so that

$$6r + 4s + 2\dim(\mathfrak{k}_{ss}) \le 2n,$$

and the second inequality is proved.

Conversely, we start from a Lie algebra $\mathfrak{k}$ satisfying the conditions of the theorem above, and we construct a $\mathfrak{k}$-equivariant map φ as in Lem. 2.3.1 which is in direct correspondence with a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and with a canonical connection $\tilde{\nabla}$ of $\mathbb{C}\mathrm{H}(n)$.

First, let $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_{ss}$ be a reductive Lie algebra of compact type which satisfies the first inequality of dimensions given in the theorem.

Since $\dim \mathfrak{k} < \dim \mathfrak{n}_1$ we can consider a subspace $V_{\mathfrak{k}} \subset \mathfrak{n}_1$ together with a linear $\mathfrak{k}$-modules isomorphims $\psi \colon V_{\mathfrak{k}} \longrightarrow \mathfrak{k}$. We transfer the decomposition $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_{ss}$ to $V_{\mathfrak{k}}$ by ψ as $V_0 + V_{ss}$. We consider V_1 a minimal effective metric and complex representation of $\mathfrak{k}_0 \cong \mathbb{C}^r \times \mathbb{R}^s$ and we take W a copy of V_{ss}. As $\dim(V_1) = 4r + 2s$, and using the constraints of dimensions we define,

$$\mathfrak{n}_1 = \mathbb{C}^{n-1} = V_{\mathfrak{k}} + V_1 + W + \mathbb{R}^s + \mathbb{C}^m$$

for certain $m \ge 0$, where $L = \mathbb{R}^s + \mathbb{C}^m$ is a trivial submodule of $\mathfrak{k}$, and we define a complex structure between the $\mathbb{R}^s$ factors of V_0 and L. This decomposition of $\mathfrak{n}_1$ admits a $\mathfrak{k}$-invariant inner product compatible with the complex structure: in $V_0 + L = \mathbb{C}^{r+s+m}$; in V_1, the inner product is induced by the effective representation; in V_{ss}, we use that it is isomorphic to a semi-simple Lie subalgebra of a compact Lie algebra. Hence, it admits a bi-invariant real inner product which can be extended to a Hermitian inner product in $V_{ss} + W$. Therefore, $\mathfrak{k}$ acts effectively in $\mathbb{C}^{n-1}$ preserving the Hermitian inner product and $\mathfrak{k}$ arises as a Lie subalgebra of

$\mathfrak{u}(n-1)$ which is isomorphic to $\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))$. Finally, we define φ to be ψ on $V_{\mathfrak{k}}$ and zero on $V_1+W+L+\mathfrak{a}+\mathfrak{n}_2$. This realises $\mathfrak{k}$ as the holonomy algebra of a canonical connection on $\mathbb{C}\mathrm{H}(n)$ with $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$.

Second, let $\mathfrak{k}=\mathfrak{k}_0+\mathfrak{k}_{ss}$ be a reductive Lie algebra of compact type which satisfies the second inequality of dimensions given in the theorem and $s\geq 1$.

We consider a copy $V_{\mathfrak{k}}\subset\mathfrak{n}$, with non-trivial projection to n_2, of the $\mathfrak{k}$-module $\mathfrak{k}$ given by a linear isomorphism $\psi\colon V_{\mathfrak{k}}\longrightarrow\mathfrak{k}$. We decompose $V_{\mathfrak{k}}$ as V_0+V_{ss} by ψ, with $V_0=\mathfrak{n}_2+V_0'$. We consider V_1 a minimal effective metric and complex representation of $\mathfrak{k}_0\cong\mathbb{C}^r\times\mathbb{R}^s$ and we take W a copy of V_{ss}. As $\dim(V_1)=4r+2s$, and using the constraints of dimensions we define,

$$\mathfrak{n}_1=\mathbb{C}^{n-1}=V_0'+V_{ss}+V_1+W+\mathbb{R}^s+\mathbb{C}^m,$$

for certain $m\geq 0$, where $L=\mathbb{C}^m+\mathbb{R}^{s-1}$ is a trivial submodule of $\mathfrak{k}$ and we define a complex structure between the $\mathbb{R}^s$ factors of V_0' and L. This decomposition of $\mathfrak{n}_1$ admits a $\mathfrak{k}$-invariant inner product compatible with the complex structure: In $V_0'+L=\mathbb{C}^{r+(s-1)+m}$ we choose any inner product compatible; in V_1 the inner product is induced by the effective representation; in V_{ss} we use that it is isomorphic to a semi-simple Lie subalgebra of a compact Lie algebra, so that, it admits a bi-invariant real inner product which can be extended to a Hermitian inner product in $V_{ss}+W$. Hence, a $\mathfrak{k}$ arises as a Lie subalgebra of $\mathfrak{u}(n-1)$ which is isomorphic to $\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1))$. Finally, we define φ to be ψ on $V_{\mathfrak{k}}$ and zero on $V_1+L+W+\mathfrak{a}$ then realises $\mathfrak{k}$ as the holonomy algebra of a canonical connection on $\mathbb{C}\mathrm{H}(n)$ with $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$.

The construction of the canonical connection associated with $\mathfrak{k}$ is in the case $A_r=A_0$ ($\mathfrak{a}_r=\mathfrak{a}$, see Sec. 2.3) the construction is analogous for $A_r=A_0+H_r$ with $H_r\notin\mathfrak{k}$. Also, note that the Lie algebra $\mathfrak{k}$ exponentiates to a closed (so compact) subgroup K of $S(U(n-1)U(1))$. $\square$

2.6 The description of homogeneous types

To complete the study of homogeneous structures in the complex hyperbolic space we characterize, in terms of the holonomy algebra of the canonical connection, when a homogeneous structure belongs to the different subspaces of the decomposition (see the expressions in Thm. A.1)

$$\mathcal{K}(V)=\mathcal{K}_1(V)+\mathcal{K}_2(V)+\mathcal{K}_3(V)+\mathcal{K}_4(V).$$

For convenience in the rest of the section, the direct sums $\mathcal{K}_i(V)+\mathcal{K}_j(V)$ and $\mathcal{K}_i(V)+\mathcal{K}_j(V)+\mathcal{K}_k(V)$ is denoted by $\mathcal{K}_{i+j}(V)$ and $\mathcal{K}_{i+j+k}(V)$ respectively.

In Thm. 2.4.2, we found a global expression for every homogeneous structure in $\mathbb{C}\mathrm{H}(n)$. We split the expression of (2.14) in four tensors,

$$
\begin{aligned}
E_1(B,C,D) &= \alpha_D g(B,C) - \alpha_C g(B,D) + \alpha_{JD} g(B,JC) - \alpha_{JC} g(B,JD),\\
E_2(B,C,D) &= \alpha_{JB} g(\tilde{X}_{JC},\tilde{X}_D),\\
E_3(B,C,D) &= g([B',C],D),\\
E_4(B,C,D) &= \alpha_B g(C_r,D).
\end{aligned}
$$

Lemma 2.6.1. *The tensor element E_1 belongs to $\mathcal{K}_{2+4}(V)$ with*

$$\theta_2 = \theta_4 = \frac{g(\tilde{A}_0,\cdot)}{g(\tilde{A}_0,\tilde{A}_0)}.$$

Lemma 2.6.2. *The tensor elements E_2,E_3,E_4 belongs to* $\ker(c_{12})$.

Proof. We consider an orthonormal basis

$$\mathcal{B}_1 = \left\{ \tilde{e}_i, \tilde{e}_n = \frac{\tilde{A}_0}{\sqrt{\mu}}, \tilde{e}_{i+n}, \tilde{e}_{2n} = \frac{-2}{\sqrt{\mu}}\tilde{N}_2 : i = 1, \dots, n-1 \right\}$$

of $\mathfrak{m}$ such that $J\tilde{e}_i = \tilde{e}_{i+n}$ and $J\tilde{e}_{i+n} = -\tilde{e}_i$. We check directly $\mathrm{c}_{12}(E_2) = 0$ and $\mathrm{c}_{12}(E_4) = 0$. We proof that $\mathrm{c}_{12}(E_3) = 0$. First, note that $[\varphi(X),\tilde{X}] = [\varphi(X),X]$, for any element $\tilde{X} \in \mathfrak{m}$. Secondly, we consider the orthogonal decomposition $\mathfrak{n} = V_{\mathfrak{k}} + V'$ (2.15) and $\mathfrak{m} = \tilde{\mathfrak{a}} + \tilde{V}_{\mathfrak{k}} + \tilde{V}'$. Therefore, we take a basis $\mathcal{B}_2 = \{\frac{\tilde{A}_0}{\sqrt{\mu}}, \tilde{e}_j : j = 2, \dots, 2n\}$ preserving the decomposition above, that is, there exists $k \in \{2, \dots, 2n\}$ such that $\tilde{e}_j \in \tilde{V}_{\mathfrak{k}}$ for any $j \in \{2, \dots, k\}$ and $\tilde{e}_j \in \tilde{V}'$ for any $j \in \{k+1, \dots, 2n\}$. From $[\varphi(A_0),\tilde{A}_0] = 0$: if $X \in V_{\mathfrak{k}}$, then it is satisfied that $[\varphi(X),X] \in V_{\mathfrak{k}} \cap V' = \{0\}$, and if $X \in V'$, then $\varphi(X) = 0$. Therefore, each term of the sum $\mathrm{c}_{12}(E_3) = g([(\tilde{e}_1)',e_1],D) + \sum_{j=1}^{k} g([(\tilde{e}_j)',e_j],D) + \sum_{j=k+1}^{2n} g([(\tilde{e}_j)',e_j],D)$ vanishes. □

Corollary 2.6.3. *Given any non-zero Kähler homogeneous structure on $\mathbb{C}\mathrm{H}(n)$ we have that*

$$\mathrm{c}_{12}(S)(D) = 4n\,\alpha_D.$$

Corollary 2.6.4. *The tensor element E_2 is of type $\mathcal{K}_{2+3+4}(V) = \mathcal{K}_2(V) + \mathcal{K}_3(V) + \mathcal{K}_4(V)$.*

Proof. The homogeneous structure $E_2 = \alpha_{JB}\,(g(JC,D) - (\alpha_{JC}\alpha_D + \eta_D\eta_{JC}))$, the first term lies in $\mathcal{K}_{2+4}(V)$ with $\theta_2 = -\theta_4 = \alpha$ and the second term lies in $\mathcal{K}_{3+4}(V)$. □

We recall the inner product in $\mathcal{K}(V)$ defined as

$$\langle S, S' \rangle = \sum_{i,j,k=1}^{2n} S_{\tilde{e}_i \tilde{e}_j \tilde{e}_k} S'_{\tilde{e}_i \tilde{e}_j \tilde{e}_k}$$

where $\mathcal{B} = \{\tilde{e}_1, \dots, \tilde{e}_{2n}\}$ is any orthonormal basis of (V, g). With respect to this inner product we have

$$\begin{aligned}\ker(c_{12})^\perp = \Big\{ S \in \mathcal{K}(V) : S_{BCD} &= g(B,C)\sigma_2(D) - g(B,D)\sigma_2(C) \\ &+ g(B,JC)\sigma_2(JD) - g(B,JD)\sigma_2(JC),\, \sigma_2 \in V^* \Big\}.\end{aligned}$$

Lemma 2.6.5. *Let S be a Kähler homogeneous structure on $\mathbb{C}\mathrm{H}(n) = G/H$ and $\mathfrak{hol}$ its holonomy algebra. Then, S belongs to $\ker(c_{12})^\perp$ if and only if $\mathfrak{hol}$ is one dimensional with $\tilde{R}_{\tilde{A}_0 \tilde{N}_2} \tilde{X} = J\tilde{X}$, for $\tilde{X} \in \tilde{\mathfrak{n}}_1$ and $\mathfrak{a}_r = \mathfrak{a} \subset \ker \varphi$.*

Proof. Since $E_1 \in \ker(c_{12})^\perp$, from Lem. 2.6.2 we have that $S \in \ker(c_{12})^\perp$ if and only if $E_2 + E_3 + E_4 = 0$, that is, for any $B, C, D \in \mathfrak{m}$,

$$g([B', C], D) + \alpha_B g(C_r, D) = -\alpha_{JB} g(\tilde{X}_{JC}, \tilde{X}_D). \tag{2.16}$$

Suppose $E_3 + E_4 = -E_2$. Then, if we take $B = \tilde{X}_B \in \tilde{\mathfrak{n}}_1$, we get $g([\varphi(X_B), C], D) = 0$, which means that $\varphi(X_B) = 0$ and $\mathfrak{hol} = \varphi(\mathfrak{n})$ is only generated by $\varphi(N_2)$. If we take $B = \tilde{A}_0$, then $g([\varphi(A_0), C], D) + g([H_r, C], D) = 0$ which means that $\varphi(A_0) = -H_r$. Nevertheless, in the beginning of Sec. 2.3 we showed, $\varphi(A_0) \in \mathfrak{h}$ and $H_r \notin \mathfrak{h}$ or $H_r = 0$ where $\mathfrak{h}$ is the Lie algebra of H. Then, necessarily $H_r = 0 = \varphi(A_0)$, indeed $\mathfrak{a}_r = \mathfrak{a} \subset \ker \varphi$. Finally, if we take $B = \tilde{N}_2$ in (2.16) we have

$$\eta_B g([\varphi(N_2), C], D) = -\alpha_{JB} g(\tilde{X}_{JC}, \tilde{X}_D)$$

and as $\varphi(N_2)$ acts trivially on $\mathfrak{a} + \mathfrak{n}_2$ and $\eta_B = 2\alpha_{JB}$, then,

$$\eta_B g([\varphi(N_2), C], D) = 2\alpha_{JB} g([\varphi(N_2), \tilde{X}_C], \tilde{X}_D) = -\alpha_{JB} g(\tilde{X}_{JC}, \tilde{X}_D).$$

Therefore, $\tilde{X}_{JC} = -2[\varphi(N_2), \tilde{X}_C] = \tilde{R}_{\tilde{A}_0 \tilde{N}_2} \tilde{X}_C$.

Conversely, it is easy to check that for that holonomy algebra conditions it is satisfied $E_2 + E_3 + E_4 = 0$. □

Lemma 2.6.6. *Let $S = E_1 + E_2 + E_3 + E_4$ be a Kähler homogeneous structure on $\mathbb{C}\mathrm{H}(n) = G/H$ and $\mathfrak{hol}$ its holonomy algebra. Then, $E_3 + E_4$ belongs to $\mathcal{L}(V) = \{S \in \mathcal{K}(V) : S_{BCD} = g(\tilde{X}_{JC}, \tilde{X}_D)\phi(JB),\ \phi \in V^*\}$ if and only if one of the following two cases occurs:*

- *The holonomy algebra $\mathfrak{hol}$ is one dimensional such that $\tilde{R}_{\tilde{A}_0\tilde{N}_2}\tilde{X} = \lambda J\tilde{X}$, for $\tilde{X} \in \tilde{\mathfrak{n}}_1$ where $\lambda \in \mathbb{R}$ with $\lambda \neq 1$, $\mathfrak{a}_r = \mathfrak{a}$ and $\varphi(\mathfrak{a}) = \varphi(\mathfrak{n}_2) = \mathfrak{hol}$.*
- *The holonomy algebra $\mathfrak{hol}$ is trivial and $[\varphi(A_0) + H_r, \tilde{X}] = \beta J\tilde{X}$, for $\tilde{X} \in \tilde{\mathfrak{n}}_1$ where $H_r = A_r - A_0$ and $\beta \in \mathbb{R}$.*

Proof. Suppose that $E_3 + E_4 \in \mathcal{L}$, that is, for any $B, C, D \in \mathfrak{m}$,

$$g([B', C], D) + \alpha_B g(C_r, D) = \phi(B) g(\tilde{X}_{JC}, \tilde{X}_D). \tag{2.17}$$

If we take $B = \tilde{X}_B \in \tilde{\mathfrak{n}}_1$, then,

$$g([\varphi(X_B), C], D) = g([\varphi(X_B), \tilde{X}_C], \tilde{X}_D) = \phi(\tilde{X}_B) g(\tilde{X}_{JC}, \tilde{X}_D),$$

where for the first equality we have used that $\mathfrak{hol}$ acts trivially in $\tilde{\mathfrak{a}} + \tilde{\mathfrak{n}}_2$. Therefore, we have that $[\varphi(X_B), \tilde{X}_C] = \phi(\tilde{X}_B)\tilde{X}_{JC}$. Nevertheless, if we take $\tilde{X}_C = \tilde{X}_B$, then for all $\tilde{X}_B \in \tilde{\mathfrak{n}}_1$, $\phi(\tilde{X}_B)\tilde{X}_{JB} = 0$. This last equation implies $\phi(\tilde{X}_B) = 0$ and $\varphi(X_B) = 0$, for all $X_B \in \mathfrak{n}_1$. Then necessarily the holonomy algebra $\mathfrak{hol} = \varphi(\mathfrak{n})$ is trivial or an one dimensional subspace generated by $\tilde{R}_{\tilde{A}_0\tilde{N}_2} = \varphi(N_2)$.

We take $B = \tilde{N}_2$ and $B = \tilde{A}_0$ in (2.17) we have

$$g([\varphi(N_2), \tilde{X}_C], \tilde{X}_D) = \phi(\tilde{N}_2) g(\tilde{X}_{JC}, \tilde{X}_D) \tag{2.18}$$

$$g([\varphi(A_0) + H_r, \tilde{X}_C], \tilde{X}_D) = \phi(\tilde{A}_0) g(\tilde{X}_{JC}, \tilde{X}_D), \tag{2.19}$$

respectively. From these two equations, we have $[\varphi(N_2), \tilde{X}_C] = \phi(\tilde{N}_2)\tilde{X}_{JC}$ and $[\varphi(A_0) + H_r, \tilde{X}_C] = \phi(\tilde{A}_0)\tilde{X}_{JC}$. Now we consider the two cases above, first, if $\mathfrak{hol}$ is zero, then $\varphi(N_2) = 0$ and $[\varphi(A_0) + H_r, \tilde{X}_C] = \beta J\tilde{X}_B$ with $\beta = \phi(\tilde{A}_0)$. Secondly, if $\mathfrak{hol}$ is non zero, then $\varphi(N_2) \neq 0$ and $[\varphi(N_2), \tilde{X}_C] = \lambda J\tilde{X}_B$ with $\lambda = \phi(\tilde{N}_2)$. Moreover, in this case, necessarily $H_r = 0$. As $\varphi(A_0), \varphi(N_2) \in \mathfrak{h}$, $H_r \notin \mathfrak{h}$ (beginning of Sec. 2.3) and because (2.18) and (2.19), $H_r = \frac{\phi(\tilde{A}_0)}{\phi(\tilde{N}_2)}\varphi(N_2) - \varphi(A_0) = 0$.

Conversely, it is direct to check that taking these two cases S belongs to $\mathcal{L}$. $\square$

Remark 2.6.7. The subspace $\mathcal{K}_{2+4}(V)\cap\ker(c_{12})$ has the expression,

$$\Big\{S\in\mathcal{K}(V):\,S_{BCD}=g(B,C)\gamma(D)-g(B,D)\gamma(C)+g(B,JC)\gamma(JD)\\ -g(B,JD)\gamma(JC)+2ng(JC,D)\gamma(JB),\,\gamma\in V^*\Big\}.$$

Lemma 2.6.8. *Let S be a Kähler homogeneous structure on $\mathbb{C}\mathrm{H}(n)=G/H$ and $\mathfrak{hol}$ its holonomy algebra. Then, S belongs to $\mathcal{K}_{1+2+4}(V)$ if and only if S belongs to $\mathcal{K}_{2+4}(V)$.*

Proof. Let $S=E_1+E_2+E_3+E_4$ be a homogeneous structure in $\mathcal{K}_{1+2+4}(V)$. As $E_1\in\ker(c_{12})^{\perp}$ and $E_2+E_3+E_4\in\ker(c_{12})$, then $S\in\mathcal{K}_{1+2+4}(V)$ if and only if $E_2+E_3+E_4\in\mathcal{K}_1(V)+\mathcal{K}_{2+4}(V)\cap\ker(c_{12})$. Equivalently, because of Rmk. 2.6.7, there exists $\gamma\in V^*$ and $S^1\in\mathcal{K}_1(V)$ such that,

$$\begin{aligned}\alpha_{JB}g(\tilde{X}_{JC},\tilde{X}_D)+g([B',C],D)+\alpha_B g(C_r,D)&=g(B,C)\gamma(D)\\ -g(B,D)\gamma(C)+g(B,JC)\gamma(JD)-g(B,JD)\gamma(JC)&\\ +2ng(JC,D)\gamma(JB)+S^1_{BCD}.&\end{aligned}\tag{2.20}$$

As $S^1\in\mathcal{K}_1(V)$ (see the expressions in Thm. A.1), then

$$S^1_{B\tilde{N}_2\tilde{A}_0}=-S^1_{\tilde{A}_0\tilde{N}_2B}+S^1_{\tilde{N}_2\tilde{A}_0B}\tag{2.21}$$

for all $B\in\mathfrak{m}$. Now we proceed by parts. Firstly, by substituting $B\in\tilde{\mathfrak{n}}_1$, $C=\tilde{N}_2$ and $D=\tilde{A}_0$ in (2.20), we get

$$n\mu\gamma(JB)+S^1_{B\tilde{N}_2\tilde{A}_0}=0.$$

Secondly, by substituting $B=\tilde{A}_0$, $C=\tilde{N}_2$ and $D\in\tilde{\mathfrak{n}}_1$ in (2.20), we get

$$\frac{\mu}{2}\gamma(JD)+S^1_{\tilde{A}_0\tilde{N}_2D}=0.$$

By substituting $B=\tilde{N}_2$, $C=\tilde{A}_0$ and $D\in\tilde{\mathfrak{n}}_1$ in (2.20), we get

$$\frac{-\mu}{2}\gamma(JD)+S^1_{\tilde{N}_2\tilde{A}_0D}=0.$$

Finally, taking in consideration these equations in (2.21), we obtain,

$$n\mu\gamma(JD)=-\mu\gamma(JD).$$

Therefore, $\gamma(JD)=0$ for any $D\in\mathfrak{n}_1$. Now we claim that $\gamma(\tilde{N}_2)=0$ and $\gamma(\tilde{A}_0)=0$. From (2.21), we get that $S^1_{\tilde{A}_0\tilde{N}_2\tilde{A}_0}=0$ and by substituting in (2.20), we conclude that $\gamma(\tilde{N}_2)=0$. Arguing analogously, we prove that $\gamma(\tilde{A}_0)=0$. This proves that $\gamma=0$ and $E_2+E_3+E_4\in\mathcal{K}_1(V)$. As $(E_2)_{BCD}$ belongs to $\mathcal{K}_{2+3+4}(V)$ and is non-zero if and only if B is co-linear with $\tilde{N}_2$, then it is necessary that $(E_2+E_3+E_4)_{\tilde{N}_2CD}=0$ for any B, $C\in\mathfrak{m}$. We compute this and get

$$\frac{1}{2}g(\tilde{X}_{JC},\tilde{X}_D)+g([\varphi(N_2),X_C],X_D)=0.$$

Then, $[\varphi(N_2),X_C]=-\frac{1}{2}JX_C$. Therefore, $\varphi(N_2)$ acts effectively in all $\mathfrak{n}_1$ and because of Lem. 2.5.1 and Lem. 2.5.2, $\mathfrak{hol}$ is generated by $\varphi(N_2)$. Indeed, this implies that $g([B',C],D)=-\alpha_{JB}g(\tilde{X}_{JC},\tilde{X}_D)$. Therefore, $\alpha_B g(C_r,D)\in\mathcal{K}_1(V)$, taking in account the identity of $\mathcal{K}_1(V)$ (see the expressions in Thm. A.1). We finish that $\alpha_B g(C_r,D)=0$ and $E_2+E_3+E_4=0$. □

Theorem 2.6.9. *Let S be a Kähler homogeneous structure on $\mathbb{C}\mathrm{H}(n)$ and $\mathfrak{hol}$ its holonomy algebra. Then,*

1. *$S=0$ if and only if $\mathfrak{hol}=\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(1))$.*
2. *S belongs strictly to $\mathcal{K}_{2+4}(V)$ if and only if $\mathfrak{hol}$ is one dimensional with $\tilde{R}_{\tilde{A}_0\tilde{N}_2}\tilde{X}=J\tilde{X}$, for $\tilde{X}\in\tilde{\mathfrak{n}}_1$, and $\mathfrak{a}_r=\mathfrak{a}\subset\ker\varphi$.*
3. *S belongs strictly to $\mathcal{K}_{2+3+4}(V)$ if and only if one of the two cases occurs:*
   - *The holonomy algebra $\mathfrak{hol}$ is one dimensional with $\tilde{R}_{\tilde{A}_0\tilde{N}_2}\tilde{X}=\lambda J\tilde{X}$, for $\tilde{X}\in\tilde{\mathfrak{n}}_1$ $\lambda\neq 1$, $\mathfrak{a}_r=\mathfrak{a}$ and $\varphi(\mathfrak{a})=\varphi(\mathfrak{n}_2)=\mathfrak{hol}$.*
   - *The holonomy algebra $\mathfrak{hol}$ is trivial and $[\varphi(A_0)+H_r,\tilde{X}]=\beta J\tilde{X}$, for $\tilde{X}\in\tilde{\mathfrak{n}}_1$ where $H_r=A_r-A_0$ and $\beta\in\mathbb{R}$.*
4. *Otherwise, S is of general type.*

Proof. We proceed by parts.

We prove the first assertion. Let S be a homogeneous structure of $\mathbb{C}\mathrm{H}(n)$, S is equal to zero if and only if the Levi-Civita connection coincides with the canonical connection. As the holonomy algebra of the Levi-Civita connection coincides with $\mathfrak{hol}$, then $\mathfrak{hol}=\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(1))$.

We prove the second assertion. Consider the decomposition $\mathcal{K}_{2+4}(V)=\ker(c_{12})^\perp+\mathcal{K}_{2+4}(V)\cap\ker(c_{12})$. As E_2,E_3,E_4 are orthogonal to $\ker(c_{12})^\perp$ and $E_1\in\ker(c_{12})^\perp$, then $S\in\mathcal{K}_{2+4}(V)$ if and only if $E_2+E_3+E_4\in\mathcal{K}_{2+4}(V)\cap\ker(c_{12})$. Because of Rmk. 2.6.7, then these exists a $\gamma\in V^*$ such that for every $B,C,D\in\mathfrak{m}$,

$$\begin{aligned}\alpha_{JB}g(\tilde{X}_{JC},\tilde{X}_D)+g([B',C],D)+\alpha_B g(C_r,D)=&\\ &g(B,C)\gamma(D)-g(B,D)\gamma(C)+g(B,JC)\gamma(JD)\\ &-g(B,JD)\gamma(JC)+2ng(JC,D)\gamma(JB).\end{aligned}$$

If we take in this equation above $C = \tilde{N}_2$ and $D = \tilde{A}_0$. Then, as $[\mathfrak{s}(\mathfrak{u}(n-1)+\mathfrak{u}(1)),\tilde{N}_2] = 0$ the first row is zero. In the second row we use the identity, $J\tilde{N}_2 = \frac{1}{2}\tilde{A}_0$ and $J\tilde{A}_0 = -2\tilde{N}_2$. Therefore,

$$0 = 2g(B,\tilde{N}_2)\gamma(\tilde{A}_0) - 2g(B,\tilde{A}_0)\gamma(\tilde{N}_2) + ng(\tilde{A}_0,\tilde{A}_0)\gamma(JB).$$

Now, $g(\tilde{A}_0,\tilde{A}_0) = \mu$ and solving for $\gamma(JB)$,

$$\gamma(JB) = \frac{2g(B,\tilde{A}_0)\gamma(\tilde{N}_2) - 2g(B,\tilde{N}_2)\gamma(\tilde{A}_0)}{n\mu}.$$

Finally, we consider the following cases: if $B \in \tilde{\mathfrak{n}}_1$, then $\gamma(JB) = 0$; else, if $B = \tilde{A}_0$ (recall that $J\tilde{A}_0 = -2\tilde{N}_2$), then $-2\gamma(\tilde{N}_2) = \frac{2}{n}\gamma(\tilde{N}_2)$; else, if $B = \tilde{N}_2$, then $\frac{1}{2}\gamma(\tilde{A}_0) = \frac{-1}{2n}\gamma(\tilde{N}_2)$. Consequently, if $n \neq -1$, then $\gamma = 0$ and this case is reduced to Lem. 2.6.5.

We prove the third assertion. Consider the decomposition $\mathcal{K}_{2+3+4}(V) = \mathcal{K}_{2+4}(V) \cap \ker(c_{12})^{\perp} + \mathcal{K}_{2+4}(V) \cap \ker(c_{12}) + \mathcal{K}_3(V)$. As E_1, $E_2 \in \mathcal{K}_{2+3+4}(V)$ and E_3, E_4 are orthogonal to $\mathcal{K}_{2+4}(V) \cap \ker(c_{12})^{\perp}$, then, $S \in \mathcal{K}_{2+3+4}(V)$ if and only if $E_3 + E_4 \in \mathcal{K}_{2+4}(V) \cap \ker(c_{12}) + \mathcal{K}_3(V)$. Therefore, there exists a $\gamma \in V^*$ and $S^3 \in \mathcal{K}_3(V)$ such that for every B, C, $D \in \mathfrak{m}$,

$$\begin{aligned} g([B',C],D) + \alpha_B g(C_r,D) = {} & g(B,C)\gamma(D) - g(B,D)\gamma(C) \\ & + g(B,JC)\gamma(JD) - g(B,JD)\gamma(JC) \\ & + 2ng(JC,D)\gamma(JB) + S^3_{BCD}. \end{aligned} \tag{2.22}$$

As $S^3 \in \mathcal{K}_3(V)$, then

$$S^3_{B\tilde{N}_2\tilde{A}_0} = S^3_{\tilde{A}_0\tilde{N}_2B} - S^3_{\tilde{N}_2\tilde{A}_0B}$$

for all $B \in \mathfrak{m}$. Arguing analogously as in Lem. 2.6.8 we obtain,

$$(n-1)\mu\gamma(JD) = 0$$

which implies that $\gamma(JD) = 0$ for all $D \in \tilde{\mathfrak{n}}_1$ (remind that if $n = 1$, then $S = E_1$, indeed $\gamma(JD)$ is necessarily zero). Therefore, $\gamma \in (\tilde{\mathfrak{a}} + \tilde{\mathfrak{n}}_2)^*$. Equivalently to (2.22), we can consider another $\hat{\gamma}$ co-linear with γ and another $\hat{S}^3 \in \mathcal{K}_3(V)$ such that,

$$g([B',C],D) + \alpha_B g(C_r,D) = \hat{\gamma}(JB)g(\tilde{X}_{JC},\tilde{X}_D) + \hat{S}^3_{BCD}. \tag{2.23}$$

The tensor element $\hat{\gamma}(JB)g(\tilde{X}_{JC},\tilde{X}_D)$ belongs to $\mathcal{K}_{2+3+4}(V)$ with projection to $\mathcal{K}_{2+4}(V)$ is equal to $g(B,C)\gamma(D) - g(B,D)\gamma(C) + g(B,JC)\gamma(JD) - g(B,JD)\gamma(JC) + 2ng(JC,D)\gamma(JB)$.

We claim $\hat{S}=0$. By taking B, C, $D\in\tilde{\mathfrak{n}}_1$ in (2.23) and $\hat{\gamma}(\tilde{\mathfrak{n}}_1)=0$,

$$\hat{S}^3_{BCD}=g([B',C],D),$$

then, $g([B',C],D)$ satisfies $\mathcal{K}_3(V)$ identity (see (2.2)), that is,

$$\begin{aligned}g([B',C],D)=&-\frac{1}{2}g([C',D],B)-\frac{1}{2}g([D',B],C)\\&-\frac{1}{2}g([(JC)',JD],B)-\frac{1}{2}g([(JD)',B],JC).\end{aligned}\tag{2.24}$$

Firstly, we consider the decomposition $\mathfrak{n}_1=V_{ss}+V_0+V'$ as in (2.15). Because of Lem. 2.5.1 and Lem. 2.5.2, $\mathfrak{k}_0$ acts effectively on V'. Consequently, if $X_B\in V_0+V'$, then $\hat{S}^3_{BCD}=0$. Therefore, from now, we consider $X_B\in V_{ss}$. Secondly, by substituting $D=[B',C]$ in (2.24) and using the identities: $[B',B]=0$, the Jacobi identity and $\hat{S}^3\cdot g=0$. Then,

$$\big|\big|[B',C]\big|\big|^2=g([C',B],[B',C])+g([(JC)',B],[B',JC]).\tag{2.25}$$

We decompose, $X_C=(X_C)_{ss}+(X_C)_0+(X_C)_{V'}$ and $X_{JC}=(X_{JC})_{ss}+(X_{JC})_0+(X_{JC})_{V'}$ with each sum belonging to each sum of $\mathfrak{n}_1=V_{ss}+V_0+V'$, respectively. As $\mathfrak{k}_0$ acts trivially on V_{ss} and $X_B\in V_{ss}$, then $[\varphi(X_C),X_B]=[\varphi((X_C)_{ss}),X_B]$ and $[\varphi(X_{JC}),X_B]=[\varphi((X_{JC})_{ss}),X_B]$. Therefore, because of $\ker(\varphi|_{V_{ss}})=\{0\}$, then $[\varphi((X_C)_{ss}),X_B]=-[\varphi(X_B),(X_C)_{ss}]=-[\varphi(X_B),X_C-(X_C)_{V'}]$ and $[\varphi((X_{JC})_{ss}),X_B]=-[\varphi(X_B),(X_{JC})-(X_{JC})_{V'}]$. Taking tildes and substituting these in (2.25),

$$\begin{aligned}3\big|\big|[B',C]\big|\big|^2&=\big|\big|[B',(\tilde{X}_C)_{V'}]\big|\big|^2+\big|\big|[B',(\tilde{X}_{JC})_{V'}]\big|\big|^2\\&\leq 2\big|\big|[B',C]\big|\big|^2\end{aligned}$$

Therefore, $\hat{S}^3_{BCD}=0$ for B, C, $D\in\tilde{\mathfrak{n}}_1$. From (2.23), for any $B,C\in\mathfrak{m}$, $\hat{S}^3_{BC\tilde{A}_0}=0$ and $\hat{S}^3_{BC\tilde{N}_2}=0$. Finally, using (2.24), we get $\hat{S}^3_{\tilde{A}_0BC}=\hat{S}^3_{\tilde{N}_2BC}=0$ and consequently $\hat{S}=0$. Indeed, (2.23) has de expression,

$$g([B',C],D)+\alpha_B g(C_r,D)=\hat{\gamma}(JB)g(\tilde{X}_{JC},\tilde{X}_D)$$

and we are in condition of Lem. 2.6.6.

Otherwise, as any homogeneous structure S has a non-zero part E_1 in $\ker(c_{12})^{\perp}\subset\mathcal{K}_{2+4}(V)$ and E_2, E_3, $E_4\in\ker(c_{12})$. Then, the two missing cases to study are S belongs strictly to $\mathcal{K}_{1+2+4}(V)$ or S is of general type. Because of Lem. 2.6.8, S must be of general type. □

Chapter 3

Reduction of homogeneous pseudo-Kähler structures by one-dimensional fibers

Homogeneous manifolds are a central object for many mathematical models of physical theories (for example, linear and degenerate homogeneous structures are related to homogeneous plane waves, cf. [CL17]). This is especially relevant when the space is also equipped with additional geometry such as contact or Kähler. Nevertheless, it is remarkable how little is known about the relationship of homogeneous structures when there is a map between homogeneous manifolds. An example of this is the reduction procedure of homogeneous structures, which was first introduced in [CL15] and where, in particular, the authors analyzed the relationship of the corresponding homogeneous structures of a fiber bundle equipped with a contact structure over a pseudo-Riemannian almost-Hermitian base manifold. In this chapter we tackle the dual case, i. e., fibrations of pseudo-Hermitian over almost contact metric manifolds.

One of the most significant cases of homogeneous structures are those of linear type, that is, homogeneous structures belonging to the class whose dimension grows linearly with the dimension of the manifold, see [CC19, Ch. 5]. Linear classes always provide, in the different geometries where they have been studied, interesting results, starting from the characterization of negative constant curvature (cf. [TV83, Thm. 5.2]) in Riemannian manifolds, to other surprising facts in other geometries (see [BGO11] for a survey). In our work, we show that if the Kähler manifold has a homogeneous structure of linear type, then the reduced homogeneous structure is of linear type. Furthermore, as the homogeneous structures of almost contact metric manifolds are related to the covariant derivative of the fundamental 2-form associated with it, we prove that the reduced manifold by a homogeneous linear structure is of type $\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_{12}$ of Chinea-González Classification (cf. [CG90]). Besides that, if the one-dimensional Lie group is proportional to the sum of the two vectors that define each projection to the subspaces

$\mathcal{K}_2 \oplus \mathcal{K}_4$ of the linear homogeneous structure, then the manifold is Sasakian. Moreover, the reduced manifold is cosymplectic if the sum is zero.

3.1 Almost contact metric manifolds and its homogeneous structures

Let (M,g) be a pseudo-Riemannian manifold with signature (r,s) and Levi-Civita connection ∇, equipped with a $(1,1)$-tensor ϕ, a vector field $\xi \neq 0$ and the 1-form η being its dual with respect to g such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad g(\phi(X),\phi(Y)) = g(X,Y) - \varepsilon\eta(X)\eta(Y)$$

where $\varepsilon = g(\xi,\xi) = \pm 1$. In this conditions, we say that (M,g,ϕ,ξ,η) is an *almost contact metric manifold* and we call the *fundamental* 2*-form associated with the contact structure* to,

$$\Phi(X,Y) = g(X,\phi Y).$$

In an analogous way to the decomposition of [GH80] for almost Hermitian manifolds, [CG90] gave a decomposition of the space of $(0,3)$-covariant tensors with the same symmetries of the covariant derivative of the fundamental form ($\nabla\Phi$) in a fixed point $p \in M$, that is,

$$\mathcal{C}(V) = \left\{ \alpha \in V^* \otimes V^* \otimes V^* : \, \alpha(x,y,z) = -\alpha(x,\phi y,\phi z) + \eta(y)\alpha(x,\xi,z) + \eta(z)\alpha(x,y,\xi), \, \forall x,y,z \in V \right\},$$

where $V = T_pM$. Recall that the group $\mathrm{U}(r,s) \times 1$ characterizes the canonical almost contact metric structure of $\mathbb{R}^{2n+1}$ defined by $\xi_0 = e_{2n+1}$ and

$$\phi_0 = \begin{pmatrix} J_0 & 0 \\ 0 & 0 \end{pmatrix},$$

J_0 being the standard complex structure of $\mathbb{R}^{2n}$. That is, $\mathrm{U}(r,s) \times 1$ is the subgroup of $O(2r+1,2s)$ or $O(2r,2s+1)$ (depending on the value of ε) stabilizing both ξ_0 and ϕ_0. Finally, the space $\mathcal{C}(V)$ decomposes into twelve $\mathrm{U}(r,s) \times 1$-irreducible and orthogonal submodules as,

$$\mathcal{C}(V) = \mathcal{C}_1(V) \oplus \cdots \oplus \mathcal{C}_{12}(V)$$

with explicit expressions given in [CG90] or Thm. A.3.

Almost contact metric homogeneous structures

Let (M,g,ϕ,ξ,η) be an almost contact metric manifold. Then, S is an *almost contact metric homogeneous structure* if and only if

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}g = 0, \quad \tilde{\nabla}\Phi = 0, \tag{3.1}$$

where $\tilde{\nabla} = \nabla - S$ and R is the curvature of ∇. The condition, $\tilde{\nabla}\Phi = 0$, implies that $\tilde{\nabla}\phi = 0$, $\tilde{\nabla}\xi = 0$ and $\tilde{\nabla}\eta = 0$. Moreover, the equation $\tilde{\nabla}\phi = 0$ is equivalent to $\nabla\phi = [S,\phi]$, but this condition cannot be easily implemented into the definition of the pointwise space of tensors $\mathcal{S}(V)$ (see (1.17)), $V = T_pM$, since the Levi-Civita connection depends on the metric and the first derivatives of the metric. However, we still can split this space of linear tensors under the group $\mathrm{U}(r,s)\times 1$. Then, $\mathcal{S}(V)$ decomposes into two mutually orthogonal $\mathrm{U}(r,s)\times 1$ submodules,

$$\mathcal{S}(V) = \mathcal{S}_+(V) \oplus \mathcal{S}_-(V).$$

with

$$\begin{aligned}
\mathcal{S}_+(V) &= \Big\{ S \in \mathcal{S}(V) : S_{X\phi Y\phi Z} = S_{XYZ} \Big\}, \\
\mathcal{S}_-(V) &= \Big\{ S \in \mathcal{S}(V) : S_{X\phi Y\phi Z} + S_{XYZ} = \eta(Y)S_{X\xi Z} + \eta(Z)S_{XY\xi} \Big\}.
\end{aligned}$$

Note that, $\mathcal{S}_-(V)$ coincides with $\mathcal{C}(V)$ of the previous page. The decomposition of $\mathcal{C}(V)$ into twelve mutually orthogonal and irreducible $\mathrm{U}(p,q)\times 1$-submodules is an orthogonal complement to the decomposition of $\mathcal{S}_+(V)$ into six, that is:

$$\begin{aligned}
\mathcal{S}_+(V) &= \mathcal{CS}_1(V) \oplus \cdots \oplus \mathcal{CS}_6(V) \\
\mathcal{S}_-(V) &= \mathcal{C}_1(V) \oplus \cdots \oplus \mathcal{C}_{12}(V)
\end{aligned}$$

where the classifications are given in Thm. A.2 and Thm. A.3, respectively. Cosymplectic manifolds are an important subcase of almost contact metric manifolds. They are characterized by the additional condition $\nabla\phi = 0$ (or $\nabla\Phi = 0$). Since $\mathcal{S}_-(V)$ encodes the symmetries of $\nabla\Phi$, a homogeneous structure S belongs to $\mathcal{S}_+(V)$ if and only if the manifold is cosymplectic.

3.2 Reduction of a homogeneous structure

Let $\pi\colon \bar{M} \longrightarrow M$ be a (left) G-principal bundle, where $\bar{M}$ is a pseudo-Riemannian manifold with metric $\bar{g}$, and the fibres are non-degenerate with respect to $\bar{g}$. Suppose that G acts on $\bar{M}$ by isometries.

Given $\bar{p}$, we consider $V_{\bar{p}}\bar{M}$ the vertical subspace at $\bar{p}$ and $H_{\bar{p}}\bar{M}$ its orthogonal complement with respect to $\bar{g}$. As G acts by isometries, the decomposition

$$T_{\bar{p}}\bar{M} = V_{\bar{p}}\bar{M} \oplus H_{\bar{p}}\bar{M}$$

is a principal G-connection. This connection ω is sometimes called a *mechanical connection* for its relevant role in some problems in Geometric Mechanics (see [MO98]). Furthermore, there is an unique pseudo-Riemannian metric g in M such that the restriction $\pi_* \colon H_{\bar{p}}\bar{M} \longrightarrow T_{\pi(\bar{p})}M$ is an isometry for every $\bar{p} \in \bar{M}$. Obviously, the metric g satisfies,

$$g(X,Y) \circ \pi = \bar{g}\left(X^H, Y^H\right) \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

where X^H and Y^H denote the horizontal lift of X and Y with respect to the mechanical connection.

Let $\bar{\nabla}$ be the Levi-Civita connection and $\bar{S} = \bar{\nabla} - \bar{\tilde{\nabla}}$ be a pseudo-Riemannian homogeneous structure on $\bar{M}$ invariant under the action of the structure group G. Assume that there is 1-form β taking values in $\mathrm{End}(\mathfrak{h})$ such that

$$\bar{\tilde{\nabla}}\omega = \beta \cdot \omega.$$

Then, by [CL15, Thm. 3.7], the reduced tensor field S on M defined by

$$S_X Y = \pi_*\left(\bar{S}_{X^H} Y^H\right), \qquad X, Y \in \mathfrak{X}(M),$$

is a pseudo-Riemannian homogeneous structure of (M,g).

3.3 Fibrations of pseudo-Hermitian manifolds over almost contact metric manifolds

Let θ be a nowhere vanishing vector field in $\bar{M}$. Around any point $\bar{p} \in \bar{M}$ there exists a coordinate system $(x^1, \ldots, x^m)$, $m = \dim(\bar{M})$, in a neighbourhood diffeomorphic to $[0,1]^m$ such that any integral curve of θ is given by $x^1 = const, \ldots, x^{m-1} = const$. The vector field is said to be *regular* if the domains can be always chosen such that any orbit of θ intersects them at most once. For regular vector fields, the orbit space M is a smooth manifold and the projection $\pi \colon \bar{M} \longrightarrow M$ a submersion (cf. [Pal57]). Furthermore, a regular vector field is said to be *strictly regular* if all the orbits are diffeomorphic. If θ is a complete strictly regular vector field, the one-parameter group G generated by θ ($G = \mathbb{R}$ or S^1) acts freely on $\bar{M}$ and $\pi \colon \bar{M} \longrightarrow M$ is a G-principal bundle. If we further assume that $\bar{M}$ is equipped with a pseudo-Riemannian metric

$\bar{g}$ such that $\bar{g}(\theta,\theta)=\pm 1$ (that is, θ is non-degenerate so that we can normalize it) and $\bar{g}$ is invariant by the group G, then the 1-form

$$\omega(v)=\varepsilon\bar{g}(\theta,v), \qquad v\in T\bar{M},$$

where $\varepsilon=\mathrm{sign}(\bar{g}(\theta,\theta))$, is a G-principal connection form in $\pi:\bar{M}\longrightarrow M$, a mechanical connection as we mentioned above.

Theorem 3.3.1. *Let $(\bar{M},\bar{g},\bar{J})$ be an almost pseudo-Hermitian manifold and let $\theta\in\mathfrak{X}(\bar{M})$ be a complete strictly regular unit vector field ($\varepsilon=\bar{g}(\theta,\theta)=\pm 1$). We consider that both $\bar{g}$ and $\bar{J}$ are invariant with respect to the one-parameter group G defined by θ. Then, the orbit space (M,g,ϕ,ξ,η) is an almost contact metric manifold, with*

$$g(X,Y)=\bar{g}\left(X^H,Y^H\right), \quad \phi X=\pi_*\left(\bar{J}X^H\right), \quad \xi=\pi_*\left(\bar{J}\theta\right), \tag{3.2}$$

for any X, $Y\in TM$, where X^H stands for the horizontal lift with respect to the mechanical connection, and η is the dual form of ξ, that is, $\eta(\cdot)=\varepsilon g(\cdot,\xi)$.

Proof. As θ acts preserving the metric and the complex structure tensor, we have that the tensors given in (3.2) are well defined. In addition, $\bar{J}\theta$ being orthogonal to θ, it is horizontal with respect to the mechanical connection and $g(\xi,\xi)=\bar{g}(\bar{J}\theta,\bar{J}\theta)=\bar{g}(\theta,\theta)=\varepsilon$. We have to check that

$$\phi^2=-Id+\eta\otimes\xi, \qquad g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y). \tag{3.3}$$

If $X\in\xi^{\perp}=\{Y\in TM: g(Y,\xi)=0\}=\ker\eta$, then $\bar{g}(X^H,\bar{J}\theta)=0$, which means that $\bar{J}(X^H)$ is an horizontal vector. Then,

$$\begin{aligned}(\phi\circ\phi)(X)&=\phi(\pi_*(\bar{J}X^H))=\\&=\pi_*((\pi_*(\bar{J}X^H))^H)=\pi_*(\bar{J}^2X^H)=-X.\end{aligned}$$

On the other hand, $\phi(\xi)=\pi_*(\bar{J}(\xi^H))=\pi_*(\bar{J}\bar{J}\theta)=-\pi_*(\theta)=0$, so that both sides of

$$(\phi\circ\phi)(\xi)=-\xi+\eta(\xi)\xi$$

vanish. The first equation of (3.3) is satisfied.

With respect to the second equation, given $X\in TM$, we denote by X' the orthogonal part of X with respect to ξ. Note that, since $\bar{g}(\bar{J}(X'^H),\theta)=-\bar{g}(X'^H,\bar{J}\theta)=g(X',\xi)=0$, the vector

$\bar{J}(X'^H)$ is horizontal. Then

$$\begin{aligned} g(\phi X, \phi Y) &= g(\phi(X' + \eta(X)\xi), \phi(Y' + \eta(Y)\xi)) = \\ &= g(\phi(X'), \phi(Y')) = \bar{g}(\bar{J}(X'^H), \bar{J}(Y'^H)) \\ &= \bar{g}(X'^H, Y'^H) = g(X', Y') \\ &= g(X,Y) - \varepsilon\eta(X)\eta(Y), \end{aligned}$$

and the proof is complete. □

Remark 3.3.2. On top of the structure on the reduced manifold provided in the previous result, it is easy to check that the Levi-Civita connection on M associated with g is characterized by the condition

$$\nabla_X Y = \pi_*(\bar{\nabla}_{X^H} Y^H), \qquad X, Y \in \mathfrak{X}(M).$$

As we mentioned above, since $\nabla\Phi$ belongs to $\mathcal{S}_-(V)$, the classification of almost contact metric manifolds in categories other than cosymplectic (Sasaki, trans-Sasaki, Kenmotsu, etc., see [CG90]) is equivalent to $\nabla\Phi$ belonging to different combinations of the irreducible subspaces $\mathcal{C}_1, \dots, \mathcal{C}_{12}$ in which $\mathcal{S}_-(V)$ decomposes. These subspaces can be organized in a coarser classification

$$\mathcal{S}_-(V) = \mathcal{S}_{-,1}(V) + \mathcal{S}_{-,0}(V),$$

as

$$\begin{aligned} \mathcal{S}_{-,1}(V) &= \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_{11}, \\ \mathcal{S}_{-,0}(V) &= \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{12}, \end{aligned}$$

characterized by

$$\begin{aligned} \mathcal{S}_{-,1}(V) &= \Big\{\alpha \in \mathcal{S}_-(V) : \alpha_{X\xi Z} = 0\} = \{\alpha \in \mathcal{S}_-(V) : \alpha_{XYZ} = -\alpha_{X\phi Y\phi Z}\Big\}, \\ \mathcal{S}_{-,0}(V) &= \Big\{\alpha \in \mathcal{S}_-(V) : \alpha_{XYZ} = \eta(Y)\alpha_{X\xi Z} + \eta(Z)\alpha_{XY\xi}\Big\}. \end{aligned}$$

Proposition 3.3.3. *In the conditions of Thm. 3.3.1, if $(\bar{M}, \bar{g}, \bar{J})$ is a Kähler manifold, then the quotient space (M, g, ϕ, ξ, η) is an almost contact manifold such that $\nabla\Phi$ belongs to the class $\mathcal{S}_{-,0}(V) = \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{12}$.*

Proof. For $X, Y, Z \in \mathfrak{X}(M)$ we have

$$
\begin{aligned}
(\nabla_X\phi)Y = \nabla_X(\phi Y) - \phi(\nabla_X Y) &= \pi_*(\bar\nabla_{X^H}(\phi Y)^H) - \pi_*(\bar J(\nabla_X Y)^H) = \\
&= \pi_*(\bar\nabla_{X^H} J(Y')^H) - \pi_*(\bar J(\bar\nabla_{X^H}Y^H - \varepsilon\bar g(\bar\nabla_{X^H}Y^H,\theta)\theta)) = \\
&= \pi_*\big(\bar\nabla_{X^H}JY^H - \varepsilon\bar\nabla_{X^H}\bar J(\bar g(Y^H,\bar J\theta)\bar J\theta) \\
&\quad -\bar J(\bar\nabla_{X^H}Y^H) + \varepsilon\bar g(\bar\nabla_{X^H}Y^H,\theta)\bar J\theta)\big)\,.
\end{aligned}
$$

Since $\bar\nabla J = 0$, the first and third terms of the last step above vanish and we get

$$
\begin{aligned}
(\nabla_X\phi)Y &= \pi_*(\varepsilon X^H(\bar g(Y^H,\bar J\theta))\theta) + \varepsilon\bar g(Y^H,\bar J\theta)\bar\nabla_{X^H}\theta + \varepsilon\bar g(\bar\nabla_{X^H}\bar JY^H,\theta)\bar J\theta) = \\
&= \eta(Y)\pi_*(\bar\nabla_{X^H}\theta) + \varepsilon\bar g(\bar\nabla_{X^H}Y^H,\theta)\xi
\end{aligned}
$$

so that

$$
g((\nabla_X\phi)Y,Z) = \eta(Y)\bar g(\bar\nabla_{X^H}\theta,Z^H) + \eta(Z)\bar g(\bar\nabla_{X^H}Y^H,\theta), \tag{3.4}
$$

for any vector field $\mathfrak{X}(M)$. In particular,

$$
\begin{aligned}
g((\nabla_X\phi)\xi,Z) &= \eta(\xi)\bar g(\bar\nabla_{X^H}\theta,Z^H) + \eta(Z)\bar g(\bar\nabla_{X^H}\xi^H,\theta) \\
&= \bar g(\bar\nabla_{X^H}\theta,Z^H) + \eta(Z)\bar g(\bar\nabla_{X^H}\bar J\theta,\theta),
\end{aligned}
$$

and

$$
\begin{aligned}
g((\nabla_X\phi)Y,\xi) &= \eta(Y)\bar g(\bar\nabla_{X^H}\theta,\xi^H) + \eta(\xi)\bar g(\bar\nabla_{X^H}Y^H,\theta) \\
&= \eta(Y)\bar g(\bar\nabla_{X^H}\theta,\bar J\theta) + \bar g(\bar\nabla_{X^H}Y^H,\theta) \\
&= -\eta(Y)\bar g(\bar\nabla_{X^H}\bar J\theta,\theta) + \bar g(\bar\nabla_{X^H}Y^H,\theta).
\end{aligned}
$$

Then

$$
\eta(Y)g((\nabla_X\phi)\xi,Z) + \eta(Z)g((\nabla_X\phi)Y,\xi) = \eta(Y)\bar g(\bar\nabla_{X^H}\theta,Z^H) + \eta(Z)\bar g(\bar\nabla_{X^H}Y^H,\theta).
$$

Comparing with (3.4) we finally get

$$
g((\nabla_X\phi)Y,Z) = \eta(Y)g((\nabla_X\phi)\xi,Z) + \eta(Z)g((\nabla_X\phi)Y,\xi),
$$

that is $\nabla\Phi$ belongs to $\mathcal{S}_{-,0}(V)$. □

3.4 Reduction of Kähler homogeneous structures

Theorem 3.4.1. *Let $(\bar{M},\bar{g},\bar{J})$ be a pseudo-Kähler manifold equipped with a pseudo-Kähler homogeneous structure $\bar{S}$ that is invariant under the group flow G of a complete strictly regular unit vector field θ. Suppose that*

$$\bar{\tilde{\tilde{\nabla}}}\theta = \beta \otimes \theta$$

where $\bar{\tilde{\tilde{\nabla}}} = \bar{\nabla} - \bar{S}$, and β is a 1-form on M. Then, the tensor field S on the orbit space $M = \bar{M}/G$ defined by

$$S_X Y = \pi_*(\bar{S}_{X^H} Y^H)$$

is a homogeneous almost contact metric structure on (M,g,ϕ,ξ,η) beloging to the class $\mathcal{S}_+(V) \oplus \mathcal{S}_{-,0}(V)$. Furthermore, the components $S_+ \in \mathcal{S}_+(V)$ and $S_{-,0} \in \mathcal{S}_{-,0}(V)$ of S are

$$\begin{aligned}(S_+)_{XYZ} &= S_{X\phi Y\phi Z}\\ (S_{-,0})_{XYZ} &= \eta(Y)S_{X\xi Z} + \eta(Z)S_{XY\xi},\end{aligned}$$

respectively.

Proof. Since the condition $\bar{\tilde{\tilde{\nabla}}}\theta = \beta \otimes \theta$ is equivalent to $\bar{\tilde{\tilde{\nabla}}}\omega = \beta \cdot \omega$, we are in the conditions explained in Sec. 3.2 so that S is a pseudo-Riemannian homogeneous structure. To show that S is an almost contact metric homogeneous structure, we have to prove that $\tilde{\nabla}\phi = 0$, where $\tilde{\nabla} = \nabla - S$. Let $X, Y \in \mathfrak{X}(M)$ be two vector fields,

$$\omega(\bar{\tilde{\tilde{\nabla}}}_{X^H} Y^H) = X^H\left(\omega(Y^H)\right) - (\bar{\tilde{\tilde{\nabla}}}_{X^H}\omega)(Y^H) = -\beta(X^H)\omega(Y^H) = 0$$

so that, $\bar{\tilde{\tilde{\nabla}}}_{X^H} Y^H$ is horizontal. Then $\bar{\tilde{\tilde{\nabla}}}_{X^H} Y^H$ projects to $\tilde{\nabla}_X Y$. Following the same steps in Prop. 3.3.3, we get,

$$g((\tilde{\nabla}_X \phi)Y,Z) = \eta(Y)\bar{g}(\bar{\tilde{\tilde{\nabla}}}_{X^H}\theta, Z^H) + \eta(Z)\bar{g}(\bar{\tilde{\tilde{\nabla}}}_{X^H} Y^H, \theta)$$

which implies that $\tilde{\nabla}\phi = 0$, taking again into consideration the fact that $\bar{\tilde{\tilde{\nabla}}}_{X^H} Y^H$ is horizontal.

Now, we decompose $Y = Y' + \eta(Y)\xi$, $Z = Z' + \eta(Z)\xi$ and we get

$$\begin{aligned}S_{XYZ} &= \bar{S}_{X^H Y^H Z^H}\\ &= \bar{S}_{X^H (Y')^H (Z')^H} + \eta(Y)\bar{S}_{X^H \bar{J}\theta (Z')^H} + \eta(Z)\bar{S}_{X^H (Y')^H \bar{J}\theta}\\ &= \bar{S}_{X^H (Y')^H (Z')^H} + \eta(Y)\bar{S}_{X^H \bar{J}\theta Z^H} + \eta(Z)\bar{S}_{X^H Y^H \bar{J}\theta}.\end{aligned}$$

Since $\bar{S}$ is a pseudo-Kähler homogeneous structure

$$\begin{aligned} S_{XYZ} &= \bar{S}_{X^H \bar{J}(Y')^H \bar{J}(Z')^H} + \eta(Y)\bar{S}_{X^H \bar{J}\theta Z^H} + \eta(Z)\bar{S}_{X^H Y^H \bar{J}\theta} \\ &= S_{X\phi Y\phi Z} + \eta(Y)S_{X\xi Z} + \eta(Z)S_{XY\xi}, \end{aligned}$$

which implies that $S \in \mathcal{S}_+(V) \oplus \mathcal{S}_{-,0}(V)$. Finally, it is a matter of direct checking that S_+ and $S_{-,0}$ in the statement satisfy $S_+ \in \mathcal{S}_+(V)$ and $S_{-,0} \in \mathcal{S}_{-,0}(V)$. □

Theorem 3.4.2. *Let $(\bar{M}, \bar{g}, \bar{J})$ be a pseudo-Kähler manifold equipped with a pseudo-Kähler homogeneous structure $\bar{S}$ invariant under the flow group G of a complete strictly regular unit vector field θ. Assume that $\bar{S}$ belongs to the class $\mathcal{K}_2(\bar{V}) + \mathcal{K}_4(\bar{V})$, parametrized by G-invariant vector fields χ_2 and χ_4. Then, the component $(S_{-,0}) \in \mathcal{S}_{-,0}(V)$ of the reduced homogeneous almost-contact metric structure S of the almost contact metric manifold $(M = \bar{M}/G, g, \phi, \xi, \eta)$ belongs to $\mathcal{C}_5(V) \oplus \mathcal{C}_6(V) \oplus \mathcal{C}_{12}(V)$ with projections*

$$\begin{aligned} (S_{-,0})_{(5)}(X,Y,Z) &= \varepsilon\omega(\chi)\left(\eta(Y)g(X,\phi Z) - \eta(Z)g(X,\phi Y)\right) \\ (S_{-,0})_{(6)}(X,Y,Z) &= \varepsilon\eta(\pi_*\chi)\left(\eta(Z)g(X,Y) - \eta(Y)g(X,Z)\right) \\ (S_{-,0})_{(12)}(X,Y,Z) &= \varepsilon\eta(X)\left(\eta(Y)g(Z,\pi_*\chi) - \eta(Z)g(Y,\pi_*\chi)\right) \end{aligned}$$

for X, Y, $Z \in V$, where $\chi = \chi_2 + \chi_4$.

Proof. In the expression

$$(S_{-,0})_{XYZ} = \eta(Y)S_{X\xi Z} + \eta(Z)S_{XY\xi} = \eta(Y)\bar{S}_{X^H \bar{J}\theta Z^H} + \eta(Z)\bar{S}_{X^H Y^H \bar{J}\theta}$$

of the component $\mathcal{S}_{-,0}(V)$ of S, we apply that $\bar{S} \in \mathcal{K}_2(\bar{V}) + \mathcal{K}_4(\bar{V})$, defined by vector fields χ_2 and χ_4 (see the expressions in Thm. A.1), and we have

$$\begin{aligned} (S_{-,0})_{XYZ} = \quad & \eta(Y)\big(\bar{g}(X^H,\bar{J}\theta)\bar{g}(Z^H,\chi) - \bar{g}(X^H,Z^H)\bar{g}(\bar{J}\theta,\chi) \\ & + \bar{g}(X^H,\bar{J}^2\theta)\bar{g}(\bar{J}Z^H,\chi) - \bar{g}(X^H,\bar{J}Z^H)\bar{g}(\bar{J}^2\theta,\chi) \\ & -2\bar{g}(\bar{J}^2\theta,Z^H)\bar{g}(\bar{J}X^H,\hat{\chi})\big) \\ & + \eta(Z)\big(\bar{g}(X^H,Y^H)\bar{g}(\bar{J}\theta,\chi) - \bar{g}(X^H,\bar{J}\theta)\bar{g}(Y^H,\chi) \\ & + \bar{g}(X^H,\bar{J}Y^H)\bar{g}(\bar{J}^2\theta,\chi) - \bar{g}(X^H,\bar{J}^2\theta)\bar{g}(\bar{J}Y^H,\chi) \\ & -2\bar{g}(\bar{J}Y^H,\bar{J}\theta)\bar{g}(\bar{J}X^H,\hat{\chi})\big), \end{aligned}$$

where $\chi = \chi_2 + \chi_4$, $\hat{\chi} = \chi_2 - \chi_4$. As $\bar{g}(X^H, \theta) = 0$ we get

$$\begin{aligned}
(S_{-,0})_{XYZ} = \quad & \eta(Y)\left(\varepsilon\eta(X)g(Z,\pi_*\chi) - \varepsilon\eta(\pi_*\chi)g(X,Z) + \varepsilon g(X,\phi Z)\omega(\chi)\right) \\
& + \eta(Z)\left(\varepsilon\eta(\pi_*\chi)g(X,Y) - \varepsilon\eta(X)g(Y,\pi_*\chi) - \varepsilon g(X,\phi Y)\omega(\chi)\right) \\
= \quad & \varepsilon\omega(\chi)\left(\eta(Y)g(X,\phi Z) - \eta(Z)g(x,\phi Y)\right) \\
& + \varepsilon\eta(\pi_*\chi)\left(\eta(Z)g(X,Y) - \eta(Y)g(X,Z)\right) \\
& + \varepsilon\eta(X)\left(\eta(Y)g(Z,\pi_*\chi) - \eta(Z)g(Y,\pi_*\phi)\right).
\end{aligned}$$

One easily checks from expression given in the Thm. A.3, that first, second and third lines of the last equality belong to $\mathcal{C}_5(V)$, $\mathcal{C}_6(V)$ and $\mathcal{C}_{12}(V)$ respectively. □

Recall that the Ambrose-Singer condition $\tilde{\nabla}\phi = 0$ is equivalent to $\nabla\phi = [S,\phi] = [S_-,\phi]$. Hence

$$\begin{aligned}
(\nabla_X\Phi)(Y,Z) = g((\nabla_X\phi)Y,Z) &= g((S_-)_X(\phi Y) - \phi((S_-)_XY),Z) \\
&= (S_-)_{X\phi YZ} + (S_-)_{XY\phi Z}.
\end{aligned}$$

If in addition S_- belongs to $\mathcal{S}_{-,0}(V)$, then

$$\begin{aligned}
(\nabla_X\Phi)(Y,Z) &= \eta(\phi Y)(S_-)_{X\xi Z} + \eta Z(S_-)_{X\phi Y\xi} + \eta(Y)(S_-)_{X\xi\phi Z} + \eta\phi Z(S_-)_{XY\xi} \\
&= \eta(Z)(S_-)_{X\phi Y\xi} + \eta(Y)(S_-)_{X\xi\phi Z}.
\end{aligned} \tag{3.5}$$

Proposition 3.4.3. *Under the conditions of Thm. 3.4.2, the covariant derivative $\nabla\Phi$ of the fundamental form of the almost-contact metric manifold (M,g,ϕ,ξ,η) belongs to $\mathcal{C}_5(V)\oplus\mathcal{C}_6(V)\oplus\mathcal{C}_{12}(V)$ with components*

$$\begin{aligned}
((\nabla_X\Phi)_{(5)})(Y,Z) &= \varepsilon\eta(\pi_*\chi)\left(\eta(Z)g(X,\phi Y) - \eta(Y)g(X,\phi Z)\right) \\
((\nabla_X\Phi)_{(6)})(Y,Z) &= \varepsilon\omega(\chi)\left(\eta(Z)g(X,Y) - \eta(Y)g(X,Z)\right) \\
((\nabla_X\Phi)_{(12)})(Y,Z) &= \varepsilon\eta(X)\left(\eta(Y)g(\phi Z,\pi_*\chi) - \eta(Z)g(\phi Y,\pi_*\chi)\right)
\end{aligned}$$

for X, Y, $Z \in \mathfrak{X}(M)$, where $\chi = \chi_2 + \chi_4$.

Moreover, the manifold (M,g,ϕ,ξ,η) is cosymplectic if and only if $\chi = 0$.

Proof. From Thm. 3.4.1, the part S_- of the reduced homogeneous structure S in $\mathcal{S}_-(V)$ belongs to $\mathcal{S}_{-,0}(V)$ and we can apply (3.5). By means of a straightforward computation, one shows that the components $(S_-)_{(5)}$, $(S_-)_{(6)}$ and $(S_-)_{(12)}$ provide the expressions $(\nabla_X\Phi)_{(6)}$, $(\nabla_X\Phi)_{(5)}$ and $(\nabla_X\Phi)_{(12)}$ in the statement, which belong to $\mathcal{C}_6(V)$, $\mathcal{C}_5(V)$ and $\mathcal{C}_{12}(V)$ respectively.

Finally, the three components of $\nabla\Phi$ vanish if and only if $\pi_*\chi = 0$ and $\omega(\chi) = 0$, that is, $\chi = 0$. □

Corollary 3.4.4. *Let $(\bar{M}, \bar{g}, \bar{J})$ be a pseudo- Kähler manifold equipped with a pseudo-Kähler homogeneous structure of the class $\mathcal{K}_2 + \mathcal{K}_4$ defined by vector fields χ_2 and χ_4. Suppose that $\chi = \chi_2 + \chi_4$ is a complete strictly regular vector field, and let G be its flow group. Then the orbit manifold $(M = \bar{M}/G, g, \phi, \xi, \eta)$ is Sasakian.*

Proof. The vector fields χ_2 and χ_4 satisfy $\tilde{\bar{\nabla}}\chi_2 = \tilde{\bar{\nabla}}\chi_4 = 0$ (cf. [BGO11]) so that $\chi = \chi_2 + \chi_4$ satisfies the conditions of Thm. 3.4.1. From Prop. 3.4.3, since $\pi_*\chi = 0$, we have that $\nabla\Phi$ belongs to the class $\mathcal{C}_6$, which is equivalent to being Sasakian ([CG90]). □

Theorem 3.4.5. *Let $(\bar{M}, \bar{g}, \bar{J})$ be a pseudo-Kähler manifold equipped with a pseudo-Kähler homogeneous structure $\bar{S}$ invariant under the flow group G of a complete strictly regular unit vector field θ. Assume that $\bar{S}$ belongs to the class $\mathcal{K}_2(\bar{V}) + \mathcal{K}_4(\bar{V})$, parametrized by G-invariant vector fields χ_2 and χ_4. Then, the component $(S_+) \in \mathcal{S}_+(V)$ of the reduced homogeneous almost-contact metric structure S of the almost contact metric manifold $(M = \bar{M}/G, g, \phi, \xi, \eta)$ belongs to $\mathcal{CS}_2(V) \oplus \mathcal{CS}_4(V) \oplus \mathcal{CS}_6(V)$, and its expression is*

$$\begin{aligned}(S_+)(X,Y,Z) = \quad & g(X,Y)g(Z,\rho) - \varepsilon\eta(X)\eta(Y)g(Z,\rho) - g(X,Z)g(Y,\rho) \\ & + \varepsilon\eta(X)\eta(Z)g(Y,\rho) + g(X,\phi Y)g(\phi Z,\rho) - g(X,\phi Z)g(\phi Y,\rho) \\ & - 2g(\phi Y,Z)g(\phi X,\hat{\rho}) + 2\varepsilon\eta(X)\omega(\hat{\chi})g(\phi Y,Z),\end{aligned}$$

for X, Y, $Z \in V$, where $\chi = \chi_2 + \chi_4$, $\hat{\chi} = \chi_2 - \chi_4$ and $\rho = \pi_*\chi - \eta(\pi_*\chi)\xi$, $\hat{\rho} = \pi_*\hat{\chi} - \eta(\pi_*\hat{\chi})\xi$.

Proof. In the expression,

$$(S_+)_{XYZ} = S_{X\phi Y\phi Z} = \bar{S}_{X\bar{J}Y'^H\bar{J}Z'^H} = \bar{S}_{XY'Z'}$$

of the component $\mathcal{S}_+(V)$ of S, we apply that $\bar{S} \in \mathcal{K}_2(\bar{V}) + \mathcal{K}_4(\bar{V})$, defined by vector fields χ_2 and χ_4 (see the expressions in Thm. A.1), and we have

$$\begin{aligned}(S_+)_{XYZ} = \quad & \bar{g}(X^H,Y'^H)\bar{g}(Z'^H,\pi_*\chi) - \bar{g}(X^H,Z'^H)\bar{g}(Y'^H,\pi_*\chi) \\ & + \bar{g}(X^H,\bar{J}Y'^H)\bar{g}(\bar{J}Z'^H,\pi_*\chi) - \bar{g}(X^H,\bar{J}Z'^H)\bar{g}(\bar{J}Y'^H,\pi_*\chi) \\ & - 2\bar{g}(\bar{J}Y'^H,Z'^H)\bar{g}(\bar{J}X^H,\pi_*\hat{\chi}) = \\ = \quad & g(X,Y')g(Z',\pi_*\chi) - g(X,Z')g(Y',\pi_*\chi) \\ & + g(X,\phi Y)g(\phi Z,\pi_*\chi) - g(X,\phi Z)g(\phi Y,\pi_*\chi) \\ & - 2g(\phi Y,Z')\left(g(\phi X,\pi_*\hat{\chi}) - \varepsilon\eta(X)\omega(\hat{\chi})\right).\end{aligned}$$

We use that $g(Z',\pi_*\chi_i) = g(Z,\rho_i)$, $g(\phi Y,\pi_*\chi_i) = g(\phi Y,\rho_i)$, $g(\phi Y,Z') = g(\phi Y,Z)$and we get

$$\begin{aligned}(S_+)_{XYZ} = &\quad g(X,Y')g(Z,\rho) - g(X,Z')g(Y,\rho)\\ &+ g(X,\phi Y)g(\phi Z,\rho) - g(X,\phi Z)g(\phi Y,\rho)\\ &- 2g(\phi Y,Z)g(\phi X,\hat{\rho}) + 2\varepsilon\eta(X)\omega(\hat{\chi})g(\phi Y,Z).\end{aligned}$$

Finally, taking into account that $g(X,Y') = g(X',Y)$, $g(X,Z') = g(X',Z)$, $X' = X-\eta(X)\xi$, we get the given expression. □

Corollary 3.4.6. *Let $(\bar{M},\bar{g},\bar{J})$ be a pseudo-Kähler manifold equipped with a pseudo-Kähler homogeneous structure $\bar{S}$ invariant under the flow group G of a complete strictly regular unit vector field θ. Assume that $\bar{S}$ belongs to the class $\mathcal{K}_2(\bar{V})+\mathcal{K}_4(\bar{V})$. Then the reduced homogeneous almost-contact metric structure S of the almost contact metric manifold (M,g,ϕ,ξ,η) is of linear type.*

Proof. This result is a consequence of Thm. 3.4.2 and Thm. 3.4.5. □

Chapter 4

The Ambrose-Singer theorem for general homogeneous manifolds

This chapter presents a generalization of the results in Sec. 1.3 to the case of homogeneous spaces in a broad sense, that is, independently of the presence of a pseudo-Riemannian metric on the manifold (see Thm. 4.1.2 below). More specifically, we here give a characterization of reductive and homogeneous spaces equipped with a structure defined by a tensor (or a set of tensors), not necessarily associated with a metric G-structure, through the existence of a complete connection satisfying certain conditions of the Ambrose-Singer type. With homogeneity, we understand that a Lie group acts transitively and leaves the tensors invariant. For the local version of the results, we can drop again the topological conditions on the manifold as well as the completeness of the connection. This will enable us to have only the so-called notion of AS-manifold. In that case, reductivity must be defined carefully (in particular, we follow some ideas in [Luj15]) and we show that every reductive locally homogeneous manifold in the broad sense can be equipped with an Ambrose-Singer connection. As a particular instance of our result, if one of the tensors is a pseudo-Riemannian metric, we recover all the traditional Ambrose-Singer theorems in the literature.

Since all these previous characterizations live in the realm of pseudo-Riemannian geometry, the manifold is always equipped with a background connection. Thus, considering the affine structure of the space of all linear connections, the AS-connections can be regarded as $(1,1)$-tensors called *homogeneous structure tensors*. From this starting point, we apply our main result to the case where the (non-necessarily metric) manifold is also endowed with an additional arbitrary linear connection. For this connection, the (local) transformations are assumed to be affine. We thus generalize homogeneous structure tensors to non metric situations. This line of thought had been followed, from an infinitesimal point of view, in [Opo98] and, recently, [BT21], where some non-metric homogeneous spaces with connection were tackled.

4.1 A generalization of the Ambrose-Singer Theorem

Let G be a Lie group acting transitively on a smooth manifold M. Choosing a point $p_0 \in M$, we can identify M with G/H where $H \subset G$ is the isotropy subgroup of p_0. Note that M need not be pseudo-Riemannian and G is not necessarily a group of isometries. The manifold is said to be reductive homogeneous if there is a Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for certain vector subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathrm{Ad}_h(\mathfrak{m}) = \mathfrak{m}$, $\forall h \in H$. In this case, the subspace $\mathfrak{m}$ can be identified with $T_{p_0}M$ through the map $\mathfrak{m} \longrightarrow T_{p_0}M$, $X \mapsto \frac{d}{dt}\Big|_{t=0} \exp(tX)p_0$.

The action of G on M naturally lifts to the frame bundle $\mathcal{L}(M)$. It is well known that there is an unique connection in $\mathcal{L}(M)$, that is, an unique linear connection $\tilde{\nabla}$ such that for every reference u at $p \in M$ and for each $X \in \mathfrak{m}$, the orbit $\exp(tX) \cdot u$ is horizontal. This is called the *canonical connection* of the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. This connection satisfies the following important result.

Proposition 4.1.1 ([CC19, Prop. 1.4.15]). *Let $M = G/H$ be a reductive homogeneous manifold equipped with the canonical connection $\tilde{\nabla}$ and let K be an invariant tensor field on M with respect to the action of G. Then $\tilde{\nabla}K = 0$.*

In this work, a n-dimensional manifold M with a *geometric structure* is understood as a manifold equipped with a tensor or a set of tensors $P_1, \ldots, P_r$, $r \in \mathbb{N}$. This definition is initially more relaxed than the classical notion of geometric structure in the literature (see for example [ML04]). More precisely, a traditional approach defines a geometric structure as a reduction of the frame bundle through a canonical model linear tensor $P_0 \in (\otimes^s(\mathbb{R}^n)^*) \otimes (\otimes^l \mathbb{R}^n)$ in $\mathbb{R}^n$. If L is its stabilizer by the natural action of $Gl(n,\mathbb{R})$ on $(\otimes^r(\mathbb{R}^n)^*) \otimes (\otimes^l \mathbb{R}^n)$, a (r,l)-tensor P on M defines a traditional geometric structure with model P_0 if the map

$$k : \mathcal{L}(M) \longrightarrow (\otimes^s(\mathbb{R}^n)^*) \otimes (\otimes^l \mathbb{R}^n)$$

defined by

$$k(u)(v_1, \ldots, v_s, \alpha_1, \ldots, \alpha_l) = P(u(v_1), \ldots, u(v_s), (u^*)^{-1}(\alpha_1), \ldots, (u^*)^{-1}(\alpha_l)),$$

takes values in the $Gl(n,\mathbb{R})$-orbit of P_0. In particular, the subset $Q = k^{-1}(P_0) \subset \mathcal{L}(M)$ is a L-reduction of the frame bundle. Essential examples of this situation cover the (pseudo-)Riemannian, Kähler, complex, symplectic or Poisson manifolds, among others. Note that some of these examples are metric, in the sense that one of the tensors P_i is a (pseudo)-Riemannian metric, but some other instances are non-metric.

Theorem 4.1.2. *Let M be a connected and simply-connected manifold and let $P_1, \dots, P_r$ be tensor fields defining a geometric structure on M. Then, the following statements are equivalent:*

1. *The manifold $M = G/H$ is reductive homogeneous with G-invariant tensors $P_1, \dots, P_r$.*
2. *The manifold M admits a complete linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}P_i = 0 \quad i = 1, \dots, r \tag{4.1}$$

where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion tensors of $\tilde{\nabla}$.

Proof. Suppose $M = G/H$ is a reductive homogeneous manifold with G-invariant tensor fields $P_1, \dots, P_r$. If $\tilde{\nabla}$ is the canonical connection associated with the reductive decomposition, it is well-known that the canonical connection leaves invariant $\tilde{R}$ and $\tilde{T}$, that is $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$. We also have $\tilde{\nabla}P_i = 0$, $i = 1, \dots, r$, from Prop. 4.1.1. The completeness of this connection comes from [KN63, Ch. X, Cor. 2.5].

Conversely, let $\tilde{\nabla}$ be a complete connection on M satisfying $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$, $\tilde{\nabla}P_i = 0$, $i = 1, \dots, r$. We fix a frame $u_0 \in \mathcal{L}(M)$. Let $(\tilde{P}(u_0) \longrightarrow M, \widetilde{Hol}(u_0))$, $\tilde{P}(u_0) \subset \mathcal{L}(M)$, be the holonomy bundle of the connection $\tilde{\nabla}$. To simplify the notation, we denote $\tilde{P}(u_0)$ by $\tilde{P}$ and the subgroup $\widetilde{Hol}(u_0)$ by $\tilde{H}$. We will denote by $\tilde{\mathfrak{h}}$ the Lie algebras of $\tilde{H}$ and H.

We now proceed by parts.

A construction of a complete distribution on $\tilde{P}$:

On the one hand, if we choose $\{A_1, \dots, A_m\}$ a basis of $\tilde{\mathfrak{h}}$, the associated fundamental vector fields $\{A_1^*, \dots, A_m^*\}$ on $\tilde{P}$ are complete. On the other hand, for the canonical basis $\{e_1, \dots e_n\}$ of $\mathbb{R}^n$, the standard vector fields on $\mathcal{L}(M)$,

$$B(e_1) = B_1, \quad \dots \quad B(e_n) = B_n,$$

are complete on $\mathcal{L}(M)$ since $\tilde{\nabla}$ is a complete connection (see Prop. 1.1.25). Recall that, for $\eta \in \mathbb{R}^n$, the standard vector field $B(\eta)$ is the only horizontal vector field on P such that $\theta(B(\eta)) = \eta$, where θ is the *soldering form* on $\mathcal{L}(M)$. Note that, since $\tilde{\nabla}$ restricts to $\tilde{P}$ and each B_i is horizontal with respect to it, these standard vector fields are tangent to $\tilde{P} \subset \mathcal{L}(M)$. Hence $\{A_1^*, \dots, A_m^*, B_1, \dots, B_n\}$ span a complete distribution on $\tilde{P}$.

The structure coefficients of the generating vectors are constant:

We have

$$[A_k^*, A_l^*] = [A_k, A_l]^*, \quad [A_k^*, B_i] = B(A_k(e_i)).$$

We now check that $[B_i, B_j]$ has constant coefficients. We denote by ω the connection form associated with $\tilde{\nabla}$. The curvature and torsion of ω are denoted by Ω and Θ, respectively. Then,

$$\Theta(B_i, B_j) = -\theta([B_i, B_j]) \in \mathbb{R}^n,$$
$$\Omega(B_i, B_j) = -\omega([B_i, B_j]) \in \tilde{\mathfrak{h}}.$$

Hence, the splitting $[B_i, B_j] = [B_i, B_j]^h + [B_i, B_j]^v$ with respect to ω can be written as

$$[B_i, B_j] = B(\theta([B_i, B_j])) + \omega([B_i, B_j])^* = -B(\Theta(B_i, B_j)) - (\Omega(B_i, B_j))^*.$$

For every horizontal vector $\overline{X} \in T_u \tilde{P}$ (see [KN63, Ch. III, Prop. 5.2]) we have

$$\overline{X}(\Theta_u(B_i, B_j)) = u^{-1}((\tilde{\nabla}_X \tilde{T})(X_i, X_j)) = 0,$$
$$\overline{X}(\Omega_u(B_i, B_j)e_k) = u^{-1}((\tilde{\nabla}_X \tilde{R})(X_i, X_j, X_k)) = 0,$$

where X, X_i, X_j, $X_k \in T_{\pi(u)}M$ are the projections of $\overline{X}$, B_i, B_j, B_k, respectively. Then $\Theta(B_i, B_j)$ and $\Omega(B_i, B_j)e_k$ are constants and hence $[B_i, B_j]$ is a combination of $\{A_1^*, \dots, A_m^*, B_1, \dots, B_n\}$ with constant coefficients.

***M* is a homogeneous space:**

Let G be the universal covering of $\tilde{P}$ and let $\rho\colon G \longrightarrow \tilde{P}$ be the covering map. The vector fields $\overline{A_k^*}$ and $\overline{B_i}$ on G projecting to A_k^* and B_i are complete and the coefficients of the brackets are constant. Hence, ([TV83, p. 10, Prop. 1.9]), given a chosen point $e \in \rho^{-1}(u_0)$, we can endow G with a structure of Lie group with neutral element e and such that the Lie algebra $\mathfrak{g}$ of G is generated by $\{(\overline{A_k^*})_e, (\overline{B_i})_e\}$. As $[A_k^*, A_l^*] = [A_k, A_l]^*$, we can consider the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ generated by $\{(\overline{A_k^*})_e\}$ and let $G_0 \subset G$ be the associated Lie subgroup to $\mathfrak{g}_0$.

Lemma 4.1.3. *The manifold M is diffeomorphic to G/G_0 and hence it is homogeneous.*

Proof. The map $\pi_1 = \pi \circ \rho\colon G \longrightarrow M$ is a fibration of M. We take its exact homotopy sequence:

$$\begin{tikzcd}
\dots \arrow[r] & \Pi_1(M,y) \arrow[r] \arrow[d, equal] & \Pi_0(\pi_1^{-1}(y), b) \arrow[r] & \Pi_0(G,b) \arrow[d, equal] \\
& 0 & & 0
\end{tikzcd}$$

where $b \in G$ and $\pi_1(b) = y$. We infer that $\Pi_0(\pi_1^{-1}(y), b) = 0$, that is, $\pi_1^{-1}(y)$ is connected. Since π_1 is continuous, we obtain that it is closed as well. Finally, by the equality $\pi_{1*}(\overline{A_k^*}) = 0$, we can deduce that the fibres are isomorphic to G_0 and closed.

We define

$$\begin{aligned} p\colon G/G_0 &\to M \\ [b] &\longmapsto \pi_1(b). \end{aligned}$$

This map p is well-defined. Indeed, if we take a fixed point $b_0 \in G_0$ and we express it as $b_0 = \exp(Y_1)\cdot \ldots \cdot\exp(Y_s)$, with $Y_1, \ldots, Y_s \in \{\pi_{1*}(\overline{A_k^*}) = 0\}$, then we have $\pi(b\cdot b_0) = \pi(b)$, for all $b \in G$. Furthermore, p is a diffeomorphism since it is bijective and its differential is a linear isomorphism at each point. The injectivity can be obtained from the fact that $\pi_1^{-1}(y)$ is isomorphic to G_0, where surjectivity is straightforward since ρ and π are both surjective. □

The structure tensors are invariant and M is reductive:

Lemma 4.1.4. *For any $a \in G$, the lift $\tilde{L}_a\colon \mathcal{L}(M) \longrightarrow \mathcal{L}(M)$ of the map*

$$\begin{aligned} L_a\colon M &\longrightarrow M \\ [b] &\longmapsto [a\cdot b] \end{aligned}$$

restricts to the reduction bundle $\tilde{L}_a\colon \tilde{P} \longrightarrow \tilde{P}$.

Proof. Let $\mathbf{L}_a$ be the left multiplication on G by $a \in G$. Note that $L_a \circ \pi_1 = \pi_1 \circ \mathbf{L}_a$. Then

$$(L_a)_* \circ \pi_{1*}(\overline{B_i})_b = \pi_{1*} \circ (\mathbf{L}_a)_*(\overline{B_i})_b = \pi_{1*}(\overline{B_i})_{ab},$$

and

$$\begin{aligned} \tilde{L}_a\rho(b) &= (L_a([b]); (L_a)_* \circ \pi_{1*}(\overline{B_1})_b, \ldots, (L_a)_* \circ \pi_{1*}(\overline{B_n})_b) \\ &= ([ab]; \pi_{1*}(\overline{B_1})_{ab}, \ldots, \pi_{1*}(\overline{B_n})_{ab}) \\ &= \rho(ab) \in \tilde{P}. \end{aligned}$$

Hence,

$$\tilde{L}_a(\rho(b)) = \rho(ab).$$

□

Since $\tilde{P}$ is included in the reduction of $\mathcal{L}(M)$ defined by the tensors $P_1, \ldots, P_r$, we have that $\tilde{L}_a$ preserves them.

On the other hand, $\tilde{P}$ is a Lie group. The action of G on $\tilde{P}$ introduced in the previous Lem. 4.1.4 is transitive, since $\tilde{L}_a(\rho(b)) = \rho(ab)$, and also effective because it is constructed by linear

transformation. In particular, the Lie algebra of $\tilde{P}$ is isomorphic to the Lie algebra of its universal covering G, through the isomorphism $(\rho_e)_*$ in the neutral element.
Finally, we have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the subspace generated by $\{B_i\}$, which clearly satisfies $[\mathfrak{g}_0, \mathfrak{m}] \subset \mathfrak{m}$. Since G_0 is connected, we have the Ad invariance of $\mathfrak{m}$, and the proof of Thm. 4.1.2 is completed. ■

Remark 4.1.5. The case with no geometric structure on M ($r = 0$) was treated in [KN69, Ch. X, Th. 2.8] or [Kow80]. There, the authors characterized connected and simply connected reductive homogeneous manifolds $M = G/H$ by the existence of a complete connection $\tilde{\nabla}$ such that $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$. Thm. 4.1.2 thus provides the generalization of this result to manifolds endowed with additional structures ($r \geq 1$).

Definition 4.1.6. Let $(M, P_1, \ldots, P_r)$ be a manifold equipped with a geometric structure defined by a set of tensors $P_1, \ldots, P_r$. A connection $\tilde{\nabla}$ is called a *generalized Ambrose-Singer connection* if it satisfies that:

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}P_i = 0, \quad i = 1, \ldots, r$$

where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion of $\tilde{\nabla}$.

For short, a generalized Ambrose-Singer connection is called an AS-connection, and the manifold M, equipped with the tensors $P_1, \ldots, P_r$, is called an AS-manifold.

We note that Thm. 4.1.2 generalizes the Ambrose-Singer Thm. 1.3.3 on Riemannian manifolds (M, g) by setting $r = 1$ and $P_1 = g$. In this case, the AS conditions (4.1), that is, $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$, $\tilde{\nabla}g = 0$, are known to be equivalent to the more classical conditions $\tilde{\nabla}R = 0$, $\tilde{\nabla}S = 0$, $\tilde{\nabla}g = 0$, where $S = \tilde{\nabla} - \nabla^{LC}$, and R is the curvature of the Levi-Civita connection ∇^{LC}. We now show that this equivalence can be analysed from a broader perspective for manifolds equipped with a fixed connection for which the transitive group action is via affine transformations. More precisely, we have the following result.

Theorem 4.1.7. *Let M be a connected and simply-connected manifold with an affine connection ∇ and let $P_1, \ldots, P_r$ be tensor fields defining a geometric structure on M. Then, the following statements are equivalent:*

1. *M is a reductive homogeneous space $M \simeq G/H$, the group G acts by affne transformations of ∇ and $P_1, \ldots, P_r$ are G-invariant.*

2. *The manifold M admits a complete linear connection $\tilde{\nabla}$ satisfying:*

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}T = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}P_i = 0 \quad i = 1, \ldots, r$$

where R and T are the curvature and torsion of ∇ and $S = \nabla - \tilde{\nabla}$.

Proof. Let $G \subset \mathrm{Aff}(M, \nabla)$ be a group acting transitively on M and preserving $P_1, \ldots, P_r$. Additionally, G preserves the tensor $S = \nabla - \tilde{\nabla}$. Hence, by Thm. 4.1.2 we have that

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}P_i = 0, \quad i = 1, \ldots, r$$

which are equivalent to

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}T = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}P_i = 0 \quad i = 1, \ldots, r$$

by the following observation,

$$T_X Y - \tilde{T}_X Y = S_X Y - S_Y X, \quad \tilde{R}_{XY} = R_{XY} + [S_X, S_Y] + S_{\tilde{T}_X Y}.$$

Conversely, by Thm. 4.1.2, we have that there exists a Lie group G preserving $S, P_1, \ldots, P_r$. Every transformation of G on M preserves S and is an affine transformation of $\tilde{\nabla}$. Hence, G preserves $S + \tilde{\nabla} = \nabla$ which means they are affine transformations of ∇. □

Remark 4.1.8. In particular, Thm. 4.1.7 covers the case of Riemannian homogeneous manifold when $r = 1$, $P_1 = g$ the metric tensor and $\nabla = \nabla^{LC}$ is the Levi-Civita connection. In that case, the preservation of g implies the affine nature of the transformations.

Definition 4.1.9. Let $(M, P_1, \ldots, P_r)$ be a manifold equipped with a geometric structure defined by a set of tensors $P_1, \ldots, P_r$ together with an affine connection ∇. A homogeneous structure is a collection $(M, P_1, \ldots, P_r, \nabla, \widetilde{\nabla})$ such that

$$\widetilde{\nabla}R = 0, \quad \widetilde{\nabla}T = 0, \quad \widetilde{\nabla}S = 0, \quad \widetilde{\nabla}P_i = 0, \quad i = 1, \ldots, r$$

where $S = \nabla - \widetilde{\nabla}$, and R and T are the curvature and torsion of ∇ respectively. For short, we call S a homogeneous structure tensor.

In particular, homogeneous structures of ∇^{LC} are the classical Riemannian homogeneous structures.

Recall that, in Rmk. 1.2.5 we consider one tensor $K = (P_1, \ldots, P_r)$ instead of a set of tensors. Consequently, in following sections we adopt this notation for a set of tensors and results are analogous.

4.2 Reductive locally homogeneous manifolds

The conditions involved in Thm. 4.1.2 are of three different types. First, there is a group of partial differential equations expressed as the vanishing of some covariant derivatives. Second, the completeness of the AS-connection. And finally, a couple of topological conditions (connectedness and simply-connectedness) of the manifold M. Connectedness is not an issue, since one usually works with connected components. With respect to simply connectedness, even though essential, it is a condition that can be implemented by working with the universal cover of the manifold, and then projecting the structures back to the original space. The projection will probably imply that the space is locally homogeneous only, but locally isomorphic to the global homogeneous cover. The completeness, however entails more delicate information since non-complete AS connections may induce locally homogeneous manifolds that are not locally isomorphic to homogeneous spaces. In the Riemannian case, we already introduced the classical result (see Thm. 1.3.6).

However, if one wants to move forward, the generalization to pseudo-Riemannian manifolds with signature implies the understanding of the notion of reductivity in the local case. That construction was recently achieved in [Luj15]. We generalize below the definition of reductive locally homogeneous manifolds with not necessarily metric structure, and we also characterize these manifolds through a transitive Lie pseudo-group and an AS-connection.

Let $\mathcal{G}$ be the Lie pseudo-group that acts transitively on (M,K) and preserves a geometric structure defined by a tensor or a set of tensors $K = (P_1, \ldots, P_r)$. That is, (M,K) is a locally homogeneous manifold, see Def. 1.2.6. In order to define a reductive locally homogeneous manifold, we have to know:

- The meaning of isotropy representation related to pseudo-groups.
- The meaning of adjoint function.

We again fix a frame $u_0 \in \mathcal{L}(M)$ over $p_0 \in M$. We define $\mathcal{G}(p_0)$ as the set of transformations for which p_0 belongs to the domain of φ and $\mathcal{G}(p_0, p_0) \subset \mathcal{G}(p_0)$ the set of transformations such that $\varphi(p_0) = p_0$. The quotient $H(p_0) = \mathcal{G}(p_0, p_0)/\sim$ with respect to the relation $\varphi \sim \psi \iff \varphi|_U = \psi|_U$ for some neighbourhood U of p_0, is a Lie group (cf. [Acc21, Ch. 1]). We say that the action of $\mathcal{G}$ on M is *effective* and *closed* if the map

$$\begin{aligned} H(p_0) &\longrightarrow \mathrm{GL}(n,\mathbb{R}) \\ \varphi &\longmapsto u_0^{-1} \circ \varphi_* \circ u_0 \end{aligned} \tag{4.2}$$

is a monomorphism and its image $\mathbf{H}(u_0)$ is closed, respectively; in particular, $\mathbf{H}(u_0)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. The morphism (4.2) will be called the isotropy representation of $\mathcal{G}$ on (M,K).

Proposition 4.2.1. *The action of $\mathcal{G}$ on M is effective if and only if for every φ, $\psi \in \mathcal{G}$ such that $\varphi(p_0) = \psi(p_0)$ and $\varphi_{*,p_0} = \psi_{*,p_0}$, we have $\varphi = \psi$ in an open neighbourhood of p_0.*

Proof. It is obvious that if we have the second condition then we have an effective action.

Conversely if φ, $\psi \in \mathcal{G}$ are such that $\varphi(p_0) = \psi(p_0)$ and $\varphi_{*,p_0} = \psi_{*,p_0}$, we have $\psi \circ \varphi^{-1} \in H(p_0)$, $\psi \circ \varphi^{-1}(p_0) = p_0$ and $(\psi \circ \varphi^{-1})_{*,p_0} = \mathrm{Id}_{T_{p_0}M}$. Then, $\psi \circ \varphi^{-1} = \mathrm{Id}_M$ in an open neighbourhood of p_0. □

We now consider

$$P(u_0) := \Big\{ \varphi_* \circ u_0 \colon \mathbb{R}^n \longrightarrow T_{\varphi(p_0)}M : \varphi \in \mathcal{G}(p_0) \Big\}. \tag{4.3}$$

This bundle is a reduction of $(\mathcal{L}(M) \longrightarrow M, \mathrm{GL}(n,\mathbb{R}))$ to the group $\mathbf{H}(u_0)$.

Proposition 4.2.2. *Let u_0, $u_1 \in \mathcal{L}(M)$ be two frames on p_0 and $p_1 \in M$ respectively, and $\psi \in \mathcal{G}$ with $\psi(p_0) = p_1$. Then,*

$$P(u_1) = P(u_0)g,$$

where g is the element in $\mathrm{GL}(n,\mathbb{R})$ such that $\psi_* u_0 = u_1 g^{-1}$.

Proof. We define the homomorphism $\sigma \colon H(p_0) \longrightarrow H(p_1)$, $\varphi \longmapsto \psi \circ \varphi \circ \psi^{-1}$. For the sake of simplicity, we also denote by $\sigma \colon \mathbf{H}(u_0) \longrightarrow \mathbf{H}(u_1)$ the induced homomorphism by the identification (4.2). It is a matter of checking that $R_g : \mathcal{L}(M) \longrightarrow \mathcal{L}(M)$ induces a principal bundle isomorphism between $P(u_0)$ and $P(u_1)$ with associated Lie group homomorphism σ. □

In particular, the groups $\mathbf{H}(u_0)$ and $\mathbf{H}(u_1)$ are always isomorphic. Because of this, we may simply write $\mathbf{H}$ for any $\mathbf{H}(u_0)$.

Given an element $\varphi \in H(p_0)$, we define

$$\begin{aligned} \mathrm{Ad}_\varphi \colon T_{u_0}P(u_0) &\longrightarrow T_{u_0}P(u_0) \\ \frac{d}{dt}\Big|_{t=0} (\varphi_t)_*(u_0) &\longmapsto \frac{d}{dt}\Big|_{t=0} (\varphi \circ \varphi_t \circ \varphi^{-1})_*(u_0) \end{aligned}$$

where $\varphi_t \in \mathcal{G}$, t belonging to certain interval $(-\varepsilon, \varepsilon)$.

Definition 4.2.3. Let (M,K) be a manifold with a geometric structure. We will say that (M,K) is *reductive locally homogeneous manifold* if there exists a Lie pseudo-group $\mathcal{G}$ acting transitively, effectively and closed on M, and we can decompose $T_{u_0}P(u_0) = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra associated with $H(p_0)$ and $\mathfrak{m}$ is a $\mathrm{Ad}(H(p_0))$-invariant subspace.

The definition depends at first sight on the chosen frame u_0. However, this dependence is not real as the following result proves.

Proposition 4.2.4. *Let u_0, $u_1 \in \mathcal{L}(M)$ be two linear frames. Then $T_{u_1}P(u_1)$ decomposes as $\mathfrak{h} + \mathfrak{m}_1$ for an $\mathrm{Ad}(H(p_1))$-invariant subspace $\mathfrak{m}_1$ if and only $T_{u_0}P(u_0)$ decomposes as $\mathfrak{h} + \mathfrak{m}_0$ for an $\mathrm{Ad}(H(p_0))$-invariant subspace $\mathfrak{m}_0$.*

Proof. Given the decomposition $T_{u_0}P(u_0) = \mathfrak{h} + \mathfrak{m}_0$ is such that $\mathrm{Ad}(H(p_0))_\varphi(\mathfrak{m}_0) \subset \mathfrak{m}_0$, we write $T_{u_1}P(u_1) = \mathfrak{h} + \mathfrak{m}_1$ with $\mathfrak{m}_1 = \Psi_*\mathfrak{m}_0$, where $\Psi = R_g \circ \psi_*$ and $\psi \in \mathcal{G}$ is such that $\psi_* u_0 = u_1 g^{-1}$ and $g \in GL(n,\mathbb{R})$. The subspace $\mathfrak{m}_1$ is $\mathrm{Ad}(H(p_1))$-invariant. Indeed, for any element $X = \frac{d}{dt}\Big|_{t=0}(\varphi_t)_*(u_0) \in \mathfrak{m}_0$ and $\varphi \in H(p_0)$, we have that

$$
\begin{aligned}
&\Psi_*(\mathrm{Ad}(H(p_0))_\varphi(X)) = \\
&= \frac{d}{dt}\Big|_{t=0} R_g \circ (\psi \circ \varphi \circ \varphi_t \circ \varphi^{-1} \circ \psi^{-1})_* \circ R_g^{-1}(R_g \circ \psi_*(u_0)) \\
&= \frac{d}{dt}\Big|_{t=0} R_g \circ (\psi \circ \varphi \circ \varphi_t \circ \varphi^{-1} \circ \psi^{-1})_* \circ R_g^{-1}(u_1) \\
&= \frac{d}{dt}\Big|_{t=0} (\psi \circ \varphi \circ \psi^{-1} \circ \psi \circ \varphi_t \circ \psi^{-1} \circ \psi \circ \varphi^{-1} \circ \psi^{-1})_*(u_1) \\
&= \mathrm{Ad}(H(p_1))_{(\psi\circ\varphi\circ\psi^{-1})}\Big(\frac{d}{dt}\Big|_{t=0}(\psi \circ \varphi_t \circ \psi^{-1})_*(u_1)\Big) \\
&= \mathrm{Ad}(H(p_1))_{\sigma(\varphi)}(\Psi_*(X)).
\end{aligned}
$$

□

Now we give a local version of Thm. 4.1.2 above. Furthermore, it provides a generalization of Tricerri's result Thm. 1.3.6.

Theorem 4.2.5. *Let (M,K) be a differentiable manifold with a geometric structure K. Then the following assertions are equivalent:*

1. *The manifold (M,K) is a reductive locally homogeneous space, associated with the Lie pseudo-group $\mathcal{G}$.*

2. *There exists a connection $\tilde{\nabla}$ such that:*

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}K = 0,$$

where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion of $\tilde{\nabla}$ respectively.

Proof. Given a Lie pseudo-group $\mathcal{G}$ acting transitively on (M,K) in a reductive fashion, let $(P \longrightarrow M, \mathbf{H})$ be the principal bundle associate to the structure of reductive locally homogeneous space as in (4.3), for a fixed frame $u_0 \in \mathcal{L}(M)$. We define a horizontal distribution D in P by $D_u = \Psi_*(\mathfrak{m})$, $\Psi = \psi_*$, for the unique $\psi \in \mathcal{G}$ such that $\psi_*(u_0) = u$, where $T_{u_0}P = \mathfrak{h} + \mathfrak{m}$ is the reductive decomposition. The distribution D is also $\mathbf{H}$-invariant, that is, given $Y = \Psi_*(X) \in D_u$, $X \in \mathfrak{m}$, we have that $(R_h)_*(Y) \in D_{u\cdot h}$, for $h \in \mathbf{H}$. Indeed, we write $X = \frac{d}{dt}\big|_{t=0}(\varphi_t)_*(u_0)$ and, by (4.2), $h = u_0^{-1} \circ \varphi_* \circ u_0$ for certain $\varphi \in H(p_0)$. Then

$$
\begin{aligned}
(R_h)_*(Y) = (R_h \circ \Psi)_*(X) &= \frac{d}{dt}\Big|_{t=0} R_h \circ \psi_* \circ (\varphi_t)_*(u_0) \\
&= \frac{d}{dt}\Big|_{t=0} \psi_* \circ (\varphi_t)_*(u_0 \circ u_0^{-1} \circ \varphi_* \circ u_0) \\
&= \frac{d}{dt}\Big|_{t=0} \psi_* \circ (\varphi_t)_* \circ \varphi_* \circ u_0 \\
&= \frac{d}{dt}\Big|_{t=0} \psi_* \circ \varphi_* \circ \varphi_*^{-1} (\varphi_t)_* \circ \varphi_* \circ u_0 \\
&= (\psi_* \circ \varphi_*)_* \operatorname{Ad}(H(p_0))_{\varphi^{-1}}(X).
\end{aligned}
$$

As $\operatorname{Ad}(H(p_0))_{\varphi^{-1}}(X) \in \mathfrak{m}$ by reductive condition, and $\psi \circ \varphi \in \mathcal{G}$ we get the invariance. This means that D can be understood as a linear connection $\tilde{\nabla}$.

We now show that

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}K = 0.$$

For p, $q \in M$, let γ be a path connecting them. The horizontal lift $\tilde{\gamma}$ with respect to $\tilde{\nabla}$ from $u \in P_p$ to $v \in P_q$, can be regarded as the parallel transport $T_pM \longrightarrow T_qM$. But since $v = \psi_* u$, for an element $\psi \in \mathcal{G}$, we have that the parallel transport is exactly ψ_*. We have that ψ_* preserves K and the connection $\tilde{\nabla}$ (and hence, its curvature and torsion) by construction. Therefore, K, $\tilde{R}$ and $\tilde{T}$ are invariant under parallel transport and their covariant derivatives vanish.

Conversely, given a linear connection $\tilde{\nabla}$ such that $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$, $\tilde{\nabla}K = 0$, let $\mathcal{G}$ be its Lie pseudo-group of local transvections. Since $\tilde{\nabla}K = 0$, the elements of $\mathcal{G}$ preserve K. Furthermore, see [KN63, Vol. I, p. 262, Cor. 7.5], $\mathcal{G}$ acts transitively.

To finish the proof we only have to show the reductive condition. We consider the holonomy reduction $(\tilde{P}(u_0) \longrightarrow M, \widetilde{Hol}(u_0))$ of the frame bundle associated with $\tilde{\nabla}$ and an element $u_0 \in \mathcal{L}(M)$. We first prove that $\mathcal{G}(p_0)$ acts transitively on $\tilde{P}(u_0)$, being $p_0 = \pi(u_0)$. Given $v \in \tilde{P}(u_0)$, there exists a horizontal curve connecting u_0 with v. The projection to M of that curve can be regarded as a parallel transport from p_0 to $q = \pi(v)$ that, in addition, preserves curvature and torsion. Hence by [KN63, Vol. I, p. 261, Thm. 7.4] there exists a local transvection $\psi \in \mathcal{G}(p_0)$ from p_0 to q such that ψ_* is that parallel transport. Therefore, $\varphi_*(u_0) = v$ and

$\mathcal{G}(p_0)$ acts transitively on $\tilde{P}(u_0)$. By construction, $P(u_0)$ (see (4.3)) coincides with $\tilde{P}(u_0)$. In particular, $\widetilde{Hol}(u_0) = \mathbf{H}(u_0)$ which is closed and the effective condition it is satisfied because Prop. 4.2.1.

Finally, if we consider $T_{u_0}\tilde{P}(u_0) = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m}$ is the horizontal distribution of $\tilde{\nabla}$ at u_0. To prove that $\mathfrak{m}$ is $\mathrm{Ad}(H(p_0))$-invariant, take $X = \frac{d}{dt}\Big|_{t=0}(\varphi_t)_*(u_0) \in \mathfrak{m}$, $\varphi \in \mathcal{G}(p_0)$ such that $\varphi(p_0) = p_0$ and $h = u_0^{-1} \circ (\varphi^{-1})_* \circ u_0 \in \mathbf{H}(u_0)$. We consider,

$$\begin{aligned}
\mathrm{Ad}(H(p_0))_\varphi(X) &= \frac{d}{dt}\Big|_{t=0} \varphi_* \circ (\varphi_t)_* \circ \varphi_*^{-1}(u_0) \\
&= \frac{d}{dt}\Big|_{t=0} \varphi_* \circ (\varphi_t)_* \circ u_0 \circ u_0^{-1} \circ \varphi_*^{-1} u_0 \\
&= \frac{d}{dt}\Big|_{t=0} R_h \circ \varphi_* \circ (\varphi_t)_*(u_0) = (R_h \circ \varphi_*)_*(X).
\end{aligned}$$

Hence, $(R_h \circ \varphi_*)_*(X)$ belongs to the horizontal distribution, because affine transvections preserve the horizontal distribution. $\square$

If we apply this last Theorem in the framework of Thm. 4.1.7 above, we get the following result.

Corollary 4.2.6. *Let (M,K) be a differentiable manifold with an affine connection ∇. Then the following assertions are equivalent:*

1. *The manifold (M,K) is a reductive locally homogeneous space, associated with a Lie pseudo-group contained in $\mathrm{Aff}_{loc}(M,\nabla)$.*
2. *There exists a connection $\tilde{\nabla}$ such that:*

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}K = 0,$$

or

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}T = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}K = 0,$$

where R,T and $\tilde{R},\tilde{T}$ are the curvature and torsion tensor of ∇ and $\tilde{\nabla}$, respectively, and $S = \nabla - \tilde{\nabla}$ is the tensor.

Definition 4.2.7. Let (M,K,∇) be a manifold endowed with a geometrical structure and an affine connection ∇. We will say that (M,K,∇) is a *reductive locally homogeneous manifold with* ∇ if it is reductive locally homogeneous associated with a Lie pseudo-group contained in $\mathrm{Aff}_{loc}(M,\nabla)$.

4.3 AS-manifolds and Homogeneous Structures

The aim of this section is to generalize the Riemannian infinitesimal constructions in Sec.1.3.1 to a general framework (dropping the metric dependence). That is, the Riemannian infinitesimal models (see (1.10)), Nomizu construction (see (1.12)) and transvection construction (see (1.13)).

In the previous section, we have proven that locally homogeneous and reductive manifolds are AS-manifolds, and vice versa. We now study AS-manifolds from an infinitesimal, or even pointwise, point of view.

Let V be a vector space of dimension n. Let

$$\tilde{R}\colon V\wedge V \longrightarrow \mathrm{End}(V), \quad \tilde{T}\colon V \longrightarrow \mathrm{End}(V),$$

be linear homomorphisms and let K be a set of linear tensors on V. We will say that $(\tilde{R},\tilde{T})$ is an *infinitesimal model associated with* K if it satisfies

$$\tilde{T}_X Y + \tilde{T}_Y X = 0, \tag{4.4}$$

$$\tilde{R}_{XY}Z + \tilde{R}_{YX}Z = 0, \tag{4.5}$$

$$\tilde{R}_{XY}\cdot\tilde{T} = \tilde{R}_{XY}\cdot\tilde{R} = 0, \tag{4.6}$$

$$\underset{XYZ}{\mathfrak{S}}\tilde{R}_{XY}Z + \tilde{T}_{\tilde{T}_X Y}Z = 0, \tag{4.7}$$

$$\underset{XYZ}{\mathfrak{S}}\tilde{R}_{\tilde{T}_X YZ} = 0, \tag{4.8}$$

$$\tilde{R}_{XY}\cdot K = 0, \tag{4.9}$$

where $\underset{XYZ}{\mathfrak{S}}$ is the cyclic sum, and $\tilde{R}_{XY}$ acts in a natural way in the tensor algebra of V as a derivation. In addition, we say that two infinitesimal models $(V,\tilde{R},\tilde{T})$ and $(V',\tilde{R}',\tilde{T}')$ are *isomorphic* if there exists a linear isomorphism $f\colon V \longrightarrow V'$ such that

$$f\tilde{R} = \tilde{R}', \quad f\tilde{T} = \tilde{T}', \quad fK = K'. \tag{4.10}$$

This notion of infinitesimal model is a generalization of the one given, see (1.10) and [Nom54; LT93].

Theorem 4.3.1. *Given a point $p_0 \in M$ of an AS-manifold $(M,K,\tilde{\nabla})$, then $(V = T_{p_0}M, \tilde{T}_{p_0}, \tilde{R}_{p_0})$ is an infinitesimal model associated with K_{p_0}, where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion of $\tilde{\nabla}$.*

Proof. Let $(M,K,\tilde{\nabla})$ be an AS-manifold. It satisfies

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}K = 0,$$

given a point $p_0 \in M$ and we recall $V = T_{p_0}M$, $\tilde{R}_0 = \tilde{R}_{p_0}$, $\tilde{T}_0 = \tilde{T}_{p_0}$ and $K_0 = K_{p_0}$, hence, $(\tilde{R}_0, \tilde{T}_0)$ is an infinitesimal model. Indeed, we deduce (4.4) and (4.5) from the skew-symmetric definition of torsion and curvature. Equations (4.6) and (4.9) come from $\tilde{\nabla}\tilde{R} = 0, \tilde{\nabla}\tilde{T} = 0, \tilde{\nabla}K = 0$. Finally, equations (4.7) and (4.8) are the Bianchi identities. □

Note that, Thm. 4.3.1 provides an infinitesimal model for every point in an AS-manifold. Now, we show it does not depend the chosen point $p_0 \in M$. Indeed,

Theorem 4.3.2. *Let $(M,K,\tilde{\nabla})$ be an AS-manifold. Given two different points p_0, $p_1 \in M$ their associated infinitesimal models are isomorphic.*

Proof. By [KN63, p. 262, Cor. 7.5], there exists a locally affine transformation φ sending p_0 to p_1. Because of being affine, we have that φ_* is a linear isomorphism between $T_{p_0}M$ and $T_{p_1}M$ satisfying $\varphi_*\tilde{T}_{p_0} = \tilde{T}_{p_1}$ and $\varphi_*\tilde{R}_{p_0} = \tilde{R}_{p_1}$. By $\tilde{\nabla}K = 0$, we conclude that $\varphi_*K_{p_0} = K_{p_1}$. □

Hence, associated with any AS-manifold there exists an unique infinitesimal model up to isomorphism. Furthermore, when different manifolds have isomorphic associated infinitesimal models, from [KN63, p. 261, Thm. 7.4] we get the following result.

Theorem 4.3.3. *Let $(M,K,\tilde{\nabla})$ and $(M',K',\tilde{\nabla}')$ be two AS-manifold and let $p_0 \in M$ and $p_0' \in M'$ be two points such that their associated infinitesimal models are isomorphic. Then there exists a local affine diffeomorphism between p_0 and p_0' sending to K to K'.*

So, we define the notion of AS-isomorphism between AS-manifolds.

Definition 4.3.4. Let $(M,K,\tilde{\nabla})$ and $(M',K',\tilde{\nabla}')$ be two AS-manifold and let $p_0 \in M$ and $p_0' \in M'$ be two points. We say $(M,K,\tilde{\nabla})$ and $(M',K',\tilde{\nabla}')$ are AS-isomorphic if there exists a local affine diffeomorphism between p_0 and p_0' sending K to K'.

From every infinitesimal model $(\tilde{R},\tilde{T})$ on V associated with K, we can construct a transitive Lie algebra using the so-called *Nomizu construction*, see (1.12). Let

$$\mathfrak{g}_0 = V \oplus \mathfrak{h}_0, \tag{4.11}$$

where $\mathfrak{h}_0 = \{A \in \mathfrak{end}(V) : A \cdot \tilde{R} = 0, A \cdot \tilde{T} = 0, A \cdot K = 0\}$, equipped with the Lie bracket

$$\begin{aligned} [A,B] &= AB - BA, && A, B \in \mathfrak{h}_0, \\ [A,X] &= AX, && A \in \mathfrak{h}_0, X \in V, \\ [X,Y] &= -\tilde{T}_X Y + \tilde{R}_{XY}, && X, Y \in V. \end{aligned} \tag{4.12}$$

Recall that, we can also consider the transvection algebra $\mathfrak{g}'_0 = V \oplus \mathfrak{h}'_0$, see (1.13), where $\mathfrak{h}'_0$ is the Lie algebra of endomorphisms generated by $\tilde{R}_{XY}$ with $X, Y \in V$, equipped with brackets as above. Then we have shown that any infinitesimal model has Nomizu and transvection constructions.

Two Nomizu constructions $(\mathfrak{g}_0, \mathfrak{h}_0, \tilde{T}, \tilde{R}, K)$ and $(\mathfrak{g}'_0, \mathfrak{h}'_0, \tilde{T}', \tilde{R}', K')$ are isomorphic if there exists a Lie algebra isomorphism $F\colon \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ such that $F(V) = V'$, F sends K to K' and $F(\mathfrak{h}_0) = \mathfrak{h}'_0$.

Proposition 4.3.5. *Two infinitesimal models are isomorphic if and only if their Nomizu constructions are isomorphic.*

Proof. Suppose that V and V' are two vector space with two infinitesimal models $(\tilde{R}, \tilde{T})$ and $(\tilde{R}', \tilde{T}')$. Then there is an isomorphism $f\colon V \longrightarrow V'$ which satisfies (4.10). We thus consider $\tilde{f}\colon \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ such that $\tilde{f}|_V = f$ and $\tilde{f}|_{\mathfrak{h}_0}(A) = f \circ A \circ f^{-1}$.

Conversely, given a Lie algebra homomorphism $F\colon \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ such that $F(V) = V'$ and $F(\mathfrak{h}_0) = \mathfrak{h}'_0$, then $f = F|_V$ is the isomorphism. Indeed, by definition f sends K to K' and, taking into account that F is a Lie algebra morphism, we obtain that f sends $\tilde{R}$ to $\tilde{R}'$ and $\tilde{T}$ to $\tilde{T}'$. □

Surprisingly, the converse is not true: two different Nomizu constructions could give rise to the same Lie algebra, see [Luj14, p. 36].

Summarizing, we have proved that there exists a morphism from the class of AS-manifolds to the class of infinitesimal models. Moreover, every infinitesimal model has associated a Nomizu construction. Now, we prove the main theorem of the section, which shows that the morphism is surjective. Note that obviously it can not be injective. The proof of this result for Riemannian manifolds can be found in [LT93].

Theorem 4.3.6. *Let V be a vector space and $(\tilde{R}_0, \tilde{T}_0)$ an infinitesimal model associated with tensors K_0. Then, there is an AS-manifold M with a geometrical structure defined by the tensor field K and a point $p_0 \in M$ such that*

$$K_{p_0} = K_0,$$

and the curvature $\tilde{R}$ and torsion $\tilde{T}$ of the AS-connection $\tilde{\nabla}$ verify that $\tilde{R}_{p_0} = \tilde{R}_0$ and $\tilde{T}_{p_0} = \tilde{T}_0$.

Any other manifold satisfying all this is locally affine diffeomorphic to $(M, \tilde{\nabla})$.

Proof. From the model $(\tilde{R}_0, \tilde{T}_0)$ on V associated with K_0, we can construct its transitive Lie algebra using the Nomizu construction, $\mathfrak{g}_0 = V \oplus \mathfrak{h}_0$ (see (4.11)). We consider a basis $\{e_1, \ldots, e_n\}$ of V and a basis $\{A_1, \ldots, A_m\}$ of $\mathfrak{h}_0$ and, respectively, we take its dual basis $\{\theta^1, \ldots, \theta^n\}$ and $\{\omega^1, \ldots, \omega^m\}$. Let G be the connected, simply connected Lie group associated with $\mathfrak{g}_0$. Indeed, the vector elements e_i and A_k and the 1-forms θ^i and ω^k of $\mathfrak{g}_0$ can be regarded as left-invariant vector fields and 1-forms on G.

Let $\phi = (x^1, \dots, x^{n+m})\colon U \subset G \longrightarrow V \subset \mathbb{R}^{n+m}$ be a local coordinate system defined on a neighbourhood U of the identity e of G such that,

$$dx^i|_e = \theta^i|_e, \quad i \in \{1, \dots, n\}. \tag{4.13}$$

We now consider the immersion map $f\colon W \subset V \longrightarrow G$ given by

$$f(y^1, \dots, y^n) = \phi^{-1}(y^1, \dots, y^n, 0, \dots, 0)$$

where W is an open neighbourhood of $0 \in \mathbb{R}^n$. We define,

$$\tilde{\theta}^i = f^*(\theta^i), \quad \tilde{e}_i = f^*(e_i), \quad i = 1, \dots, n.$$

Because of (4.13), the 1-forms $\tilde{\theta}^1, \dots, \tilde{\theta}^n$ are linearly independent at 0. Then, let $M \subset W$ be the open neighbourhood of 0 such that $\tilde{\theta}^1, \dots, \tilde{\theta}^n$ are linearly independent at each point of M.

We consider $\omega^i_j = \sum_{k=1}^m \theta^i(A_k(e_j))\omega^k$ and $\tilde{\omega}^i_j$ its pull back to M. Let $\tilde{\nabla}$ be the linear connection whose connection form is $\tilde{\omega} = (\tilde{\omega}^i_j)$. We now consider the pull-back to M of the extension $\tilde{R}$, $\tilde{T}$ and K of the tensor elements $\tilde{R}_0$, $\tilde{T}_0$ and K_0 to the Lie group G. Note that $\tilde{\omega}$ takes values in $\mathfrak{h}_0$. Therefore, by definition of $\mathfrak{h}_0$, we have $\omega(X) \cdot \tilde{R} = 0$, $\omega(X) \cdot \tilde{T} = 0$ and $\omega(X) \cdot K = 0$. Indeed, this last identities mean that $\tilde{\nabla}$ makes parallel $\tilde{R}$, $\tilde{T}$ and K.

To end, we need to prove that $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion tensors of $\tilde{\nabla}$. First, applying the definition of the exterior differential and (4.12), we have that,

$$d\theta^i + \omega^i_j \wedge \theta^j = \frac{1}{2} \sum_{j,k=1}^{n} \theta^i((\tilde{T}_0)(e_j, e_h))\theta^j \wedge \theta^h$$
$$d\omega^i_j + \sum_{k=1}^{n} \omega^i_k \wedge \omega^k_j = \frac{1}{2} \sum_{h,k=1}^{n} \theta^k((\tilde{R}_0)(e_i, e_j, e_h))\theta^h \wedge \theta^k.$$

If we pull-back these two equations to M, they are the two structural equations for the connection form ω. Therefore, the curvature and torsion of $\tilde{\nabla}$ are the tensors whose components in the basis $\{\theta^1, \dots, \theta^n\}$ are constants $\tilde{\theta}^i((\tilde{T}_0)(\tilde{e}_j, \tilde{e}_h))$ and $\theta^k((\tilde{R}_0)(\tilde{e}_i, \tilde{e}_j, \tilde{e}_h))$, respectively. Finally, the curvature and torsion of $\tilde{\nabla}$ coincide with $\tilde{R}$ and $\tilde{T}$. $\square$

We finally consider the particular case where M is a manifold with a geometric structure K equipped with a connection ∇ and an AS-connection $\tilde{\nabla}$ such that,

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}K = 0,$$

where $S = \nabla - \tilde{\nabla}$. So we can consider,

$$T_X Y = \tilde{T}_X Y + S_X Y - S_Y X, \quad R_{XY} = \tilde{R}_{XY} + [S_Y, S_X] - S_{\tilde{T}_X Y}.$$

where R and T are the curvature and torsion of ∇.

Corollary 4.3.7. *Let $(M,S,K,\tilde{\nabla})$ and $(M',S',K',\tilde{\nabla}')$ be two AS-manifolds with homogeneous structures S and S'. Then, there exists an AS-isomorphism between M and M' if and only if there exists an affine local diffeomorphism between (M,∇) and (M',∇') sending S to S' and K to K'.*

Given a fixed point $p_0 \in M$, by Thm. 4.3.1, we can consider an infinitesimal model $(T_{p_0}M, \tilde{R}_{p_0}, \tilde{T}_{p_0})$ associated with S_{p_0} and K_{p_0} with

$$\begin{aligned}
(T_{p_0})_X Y &= (\tilde{T}_{p_0})_X Y + (S_{p_0})_X Y - (S_{p_0})_Y X, \\
(R_{p_0})_{XY} &= (\tilde{R}_{p_0})_{XY} + \left[(S_{p_0})_{(}S_{p_0})_X\right] + (S_{p_0})_{(\tilde{T}_{p_0})_X Y},
\end{aligned}$$

where R_{p_0} and T_{p_0} are the curvature and torsion of ∇ in p_0.

Corollary 4.3.8. *Let $(V,\tilde{R},\tilde{T})$ and $(V',\tilde{R}',\tilde{T}')$ be two infinitesimal models associated with S, K and S', K', respectively, with*

$$\begin{aligned}
T_X Y &= \tilde{T}_X Y + S_X Y - S_Y X, \quad R_{XY} = \tilde{R}_{XY} + [S_Y, S_X] - S_{\tilde{T}_X Y}, \\
T'_X Y &= \tilde{T}'_X Y + S'_X Y - S'_Y X, \quad R'_{XY} = \tilde{R}'_{XY} + [S'_Y, S'_X] - S'_{\tilde{T}'_X Y}.
\end{aligned}$$

Hence, there exists an isomorphism of infinitesimal models if and only if there exists a linear isomorphism $f\colon V \longrightarrow V'$ such that,

$$f R = R', \quad f T = T' \quad f S = S', \quad f K = K'. \tag{4.14}$$

Chapter 5

Symplectic AS-manifolds

As we have explained, see 1.3.2, one major advantage of homogeneous structure tensors is the geometric information that one can get from their classification (we refer again to [BGO11; CC19] for surveys collecting the main contributions on this topic). Now we intend to apply similar ideas to the constructions of the previous chapter on general homogeneous manifolds, with classifications of the torsion of the Ambrose-Singer connection or, whenever there is another background connection, the corresponding homogeneous structure tensors. The achievement of these classifications shape an ambitious project to be developed in future works (almost complex, complex, contact, Poisson, etc). To begin with, in this chapter, we start this program in the case of (almost) symplectic manifolds, where explicit expressions of the classes of torsion are given. Furthermore, if the manifold is Fedosov (it has a symplectic background connection) the classification is given for homogeneous structure tensors. The relationship between both points of view is analyzed. This instance is purely non-metric and the classical results of [Kir80] cannot be applied, as we show below.

In all the known cases in the literature, the classifications of homogeneous structure tensors have some classes whose dimension grows linearly with respect to the dimension of the manifold. As we have already said, these are the so-called *classes of linear type*, and they usually provide exciting geometric characterizations (see, again, [BGO11; CC19]). Taking inspiration from these facts, we tackle the study of classes of linear type of Fedosov manifolds in the last section of this chapter. In this case, again, the geometric result is remarkable as it characterizes Hamiltonian (locally) homogeneous foliations of leaves which are flat with respect to the symplectic background connection. We provide two low-dimensional examples, leaving the study of higher-dimensional homogeneous Fedosov manifolds for future research.

5.1 Symplectic manifolds and Fedosov manifolds

An *almost symplectic manifold* (M,ω) is a differentiable manifold M equipped with a non-degenerate 2-form, ω. Additionally, if ω is closed, then it is called a *symplectic manifold*.

As it is well known (see [AP15, Thm. 2.1]), the closeness condition is equivalent to the existence of a torsion free and symplectic connection ∇, that is,

$$\nabla\omega = 0, \quad T = 0,$$

where T is the torsion of ∇. Nevertheless, this connection is not necessarily unique as it is the Levi-Civita connection. Finally, a *Fedosov manifold* (M,ω,∇) is a symplectic manifold with a torsion free and symplectic connection.

5.2 Invariant $\mathrm{Sp}(V,\omega)$-submodules of $S^2V^*\otimes V^*$ and $\wedge^2V^*\otimes V^*$

Let (V,ω) be a symplectic vector space. Based on the classifications given in [AP15], we give below explicit expressions of the invariant $\mathrm{Sp}(V)$-submodules of $S^2V^*\otimes V^*$ and $\wedge^2V^*\otimes V^*$. For that, we identify a symplectic vector space V and its dual V^* by

$$\begin{aligned}(\cdot)^*\colon V &\longrightarrow V^*\\ X &\longmapsto X^*(Y) = \omega(X,Y).\end{aligned}$$

Furthermore, we can transfer the symplectic form to V^* as $\omega^*(X^*,Y^*) = \omega(X,Y)$, that is, we regard (V,ω) and (V^*,ω^*) as symplectomorphic.

For the sake of simplicity, from now on, we denote $\omega_{XY} = \omega(X,Y)$.

Theorem 5.2.1. *If $n \geq 2$, the space of cotorsion-like tensors has the decomposition in irreducible* $\mathrm{Sp}(V)$*-submodules as*

$$S^2V^*\otimes V^* = \mathcal{S}_1(V) + \mathcal{S}_2(V) + \mathcal{S}_3(V)$$

where,

$$\begin{aligned}
\mathcal{S}_1(V) &= \left\{ S\in S^2V^*\otimes V^* : S_{XYZ} = \omega_{ZY}\omega_{XU} + \omega_{ZX}\omega_{YU},\, U\in V \right\},\\
\mathcal{S}_2(V) &= \left\{ S\in S^2V^*\otimes V^* : \mathop{\mathfrak{S}}_{XYZ} S_{XYZ} = 0,\, \mathrm{s}_{13}(S) = 0 \right\},\\
\mathcal{S}_3(V) &= \left\{ S\in S^2V^*\otimes V^* : S_{XYZ} = S_{XZY} \right\} = S^3V^*,
\end{aligned}$$

and

$$\mathrm{s}_{13}(S)(Z) = \sum_{i=1}^{n} \left(S_{e_i Z e_{i+n}} - S_{e_{i+n} Z e_i} \right),$$

for a symplectic base $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$*. If* $n = 1$*, then* $S^2V^* \otimes V^* = \mathcal{S}_1(V) + \mathcal{S}_3(V)$*.*

The dimensions of the subspaces are

$$\dim(\mathcal{S}_1(V)) = 2n, \quad \dim(\mathcal{S}_2(V)) = \frac{8}{3}(n^3 - n), \quad \dim(\mathcal{S}_3(V)) = \binom{2n+2}{3}.$$

Proof. Given a symplectic basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ of V, we define the morphisms

$$\begin{aligned}
\varphi \colon S^2V^* \otimes V^* &\longrightarrow V^* \\
(u_1^* u_2^* \otimes v^*) &\longmapsto \omega_{u_1,v} u_2^* + \omega_{u_2,v} u_1^*, \\
\pi \colon S^2V^* \otimes V^* &\longrightarrow S^3V^* \\
(u_1^* u_2^* \otimes u^*) &\longmapsto \frac{1}{3} u^* u_1^* u_2^*,
\end{aligned}$$

and

$$\begin{aligned}
\xi : V^* &\longrightarrow S^2V^* \otimes V^* \\
u^* &\longmapsto \frac{1}{2n+1} \sum_{i=1}^{n} e_i^* u^* \otimes e_{i+n}^* - e_{i+n}^* u^* \otimes e_i^*.
\end{aligned}$$

By [AP15, Thm. 1.1], applied to (V^*, ω^*), we decompose

$$S^2V^* \otimes V^* = S^3V^* + \mathcal{A}' + V^*$$

where $\mathcal{A}' = \ker(\varphi) \cap \ker(\pi)$ and V^* is isomorphic to $\mathrm{im}(\xi)$. We define $\mathcal{S}_1(V) := V^*$, $\mathcal{S}_2(V) := \mathcal{A}'$ and $\mathcal{S}_3(V) := S^3V^*$.

For the explicit expression of $\mathcal{S}_1(V)$, given $W^* \in V^*$ we have

$$\begin{aligned}
\xi(W^*)_{XYZ} &= \frac{1}{2n+1} \left(\sum_{i=1}^{n} x_{i+n} \omega_{WY} z_i + y_{i+n} \omega_{WX} z_i - x_i \omega_{WY} z_{i+n} - y_i \omega_{WX} z_{i+n} \right) \\
&= \frac{1}{2n+1} \left(\omega_{ZX} \omega_{WY} + \omega_{ZY} \omega_{WX} \right).
\end{aligned}$$

Hence, taking $U = \frac{1}{2n+1} W$, we get the required result for $\mathcal{S}_1(V)$.

With respect to the explicit expressions of $\mathcal{S}_2(V)$, for

$$S=\frac{1}{2}\sum_{i,j,k=1}^{2n} S_{e_ie_je_k}e_i^*e_j^*\otimes e_k^*\in S^2V^*\otimes V^*,$$

we have

$$\begin{aligned}
\varphi(S) &= \frac{1}{2}\sum_{i,j,k=1}^{2n} S_{e_ie_je_k}\varphi\left(e_i^*e_j^*\otimes e_k^*\right)\\
&= \frac{1}{2}\sum_{i,j,k=1}^{2n} S_{e_ie_je_k}\left(\omega_{e_ie_k}e_j^*+\omega_{e_je_k}e_i^*\right)\\
&= \sum_{i,j,k=1}^{2n}\frac{1}{2}\left(S_{e_ie_je_k}+S_{e_je_ie_k}\right)\omega_{e_ie_k}e_j^*\\
&= \sum_{j=1}^{2n}\sum_{i=1}^{n}\frac{1}{2}\left(S_{e_ie_je_{i+n}}+S_{e_je_ie_{i+n}}-S_{e_{i+n}e_je_i}-S_{e_je_{i+n}e_i}\right)e_j^*\\
&= \sum_{j=1}^{2n}\sum_{i=1}^{n}\left(S_{e_ie_je_{i+n}}-S_{e_{i+n}e_je_i}\right)e_j^*.
\end{aligned}$$

Hence, $S\in\ker\varphi$ if and only if $\mathrm{s}_{13}(S)=0$ as in the statement. Moreover, $\frac{1}{3}e_i^*e_j^*e_k^*=e_i^*e_j^*\otimes e_k^*+e_k^*e_i^*\otimes e_j^*+e_j^*e_k^*\otimes e_i^*$ and therefore

$$\pi(S)_{XYZ}=\mathop{\mathfrak{S}}_{XYZ}S_{XYZ},$$

so that we have the expression for the tensors in $\mathcal{S}_2(V)$. The dimensions come from [AP15, Thm. 1.1]. $\square$

Now, using these expressions we are going to give the explicit classes of torsion-like tensors.

Theorem 5.2.2. *If $n\geq 3$, the space of torsion-like tensors has the decomposition in irreducible* $\mathrm{Sp}(V)$*-submodules as*

$$\wedge^2V^*\otimes V^*=\tilde{\mathcal{T}}_1(V)+\tilde{\mathcal{T}}_2(V)+\tilde{\mathcal{T}}_3(V)+\tilde{\mathcal{T}}_4(V)$$

where

$$\begin{aligned}
\tilde{\mathcal{T}}_1(V) &= \left\{ \tilde{T} \in \wedge^2 V^* \otimes V^* : \tilde{T}_{XYZ} = 2\omega_{XY}\omega_{ZU} + \omega_{XZ}\omega_{YU} - \omega_{YZ}\omega_{XU},\ U \in V \right\},\\
\tilde{\mathcal{T}}_2(V) &= \left\{ \tilde{T} \in \wedge^2 V^* \otimes V^* : \mathop{\mathfrak{S}}_{XYZ} \tilde{T}_{XYZ} = 0,\ \mathrm{t}_{12}(\tilde{T}) = 0 \right\},\\
\tilde{\mathcal{T}}_3(V) &= \left\{ \tilde{T} \in \wedge^2 V^* \otimes V^* : \tilde{T}_{XYZ} = \omega_{XY}\omega_{UZ} + \omega_{YZ}\omega_{UX} + \omega_{ZX}\omega_{UY}, U \in V \right\},\\
\tilde{\mathcal{T}}_4(V) &= \left\{ \tilde{T} \in \wedge^2 V^* \otimes V^* : \tilde{T}_{XYZ} = -\tilde{T}_{XZY}, \mathrm{t}_{12}(\tilde{T}) = 0 \right\},
\end{aligned}$$

and

$$\mathrm{t}_{12}(\tilde{T})(Z) = \sum_{i=1}^{n} \tilde{T}_{e_i e_{i+n} Z},$$

for a symplectic basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$*. If* $n = 2$*, then* $\wedge^2 V^* \otimes V^* = \tilde{\mathcal{T}}_1(V) + \tilde{\mathcal{T}}_2(V) + \tilde{\mathcal{T}}_4(V)$*. If* $n = 1$*, then* $\wedge^2 V^* \otimes V^* = \tilde{\mathcal{T}}_1(V)$*.*

In addition,

$$\dim(\tilde{\mathcal{T}}_1(V)) = \dim(\tilde{\mathcal{T}}_3(V)) = 2n, \quad \dim(\tilde{\mathcal{T}}_2(V)) = \frac{8}{3}(n^3 - n), \quad \dim(\tilde{\mathcal{T}}_4(V)) = \frac{2}{3}n(2n^2 - 3n - 2).$$

Proof. For a symplectic basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ of V, we define the morphisms

$$\begin{aligned}
A_2\colon S^2V^* \otimes V^* &\longrightarrow \wedge^2 V^* \otimes V^*\\
(u_1^* u_2^* \otimes v^*) &\longmapsto v^* \wedge u_1^* \otimes u_2^* + v^* \wedge u_2^* \otimes u_1^*,\\
C\colon \wedge^2 V^* \otimes V^* &\longrightarrow V^*\\
(u_1^* \wedge u_2^* \otimes v^*) &\longmapsto \omega_{u_1 u_2} v^* + \omega_{v u_1} u_2^* + \omega_{u_2 v} u_1^*,
\end{aligned}$$

and

$$\begin{aligned}
\eta\colon V^* &\longrightarrow \wedge^3 V^*\\
u^* &\longmapsto \sum_{i=1}^{n} e_i^* \wedge e_{i+n}^* \wedge u^*.
\end{aligned}$$

By [AP15, Thm. 1.2], applied to (V^*, ω^*), we decompose

$$\wedge^2 V^* \otimes V^* = V_1^* + \mathcal{A}' + V_2^* + \mathcal{T}'$$

where, $V_1^* = A_2(\mathcal{S}_1(V))$, $\mathcal{A}' = A_2(\mathcal{S}_2(V))$, $\mathcal{T}' = \ker C \cap \wedge^3 V^*$ and $V_2^* = \mathrm{Im}(\eta)$ is the vector space such that $V_2^* \subset \wedge^3 V^*$ and $V_2^* + \mathcal{T}' = \wedge^3 V^*$. We define $\tilde{\mathcal{T}}_1(V) := V_1^*$, $\tilde{\mathcal{T}}_2(V) := \mathcal{A}'$, $\tilde{\mathcal{T}}_3(V) := V_2^*$ and $\tilde{\mathcal{T}}_4(V) := \mathcal{T}'$.

First, as

$$A_2(S)_{XYZ} = S_{YZX} - S_{XZY}, \tag{5.1}$$

we get the expression for the tensors in $\tilde{\mathcal{T}}_1(V)$ in view of the expression of $\mathcal{S}_1(V)$ in Thm. 5.2.1.

Indeed, by equation (5.1), we infer the explicit expression of $\tilde{\mathcal{T}}_1(V)$.

To study the explicit expression of $\mathcal{T}_2(V)$, we have to consider the following exact sequence, [AP15, Eq. (1.3)],

$$0 \longrightarrow S^3V^* \xrightarrow{A_1} S^2V^* \otimes V^* \xrightarrow{A_2} \wedge^2 V^* \otimes V^* \xrightarrow{A_3} \wedge^3 V^* \longrightarrow 0$$

where $A_1 = \pi$ and $A_3(u_1^* \wedge u_2^* \otimes v^*) = u_1^* \wedge u_2^* \wedge v^*$. Note that, $u_1^* \wedge u_2^* \wedge v^* = u_1^* \wedge u_2^* \otimes v^* + v^* \wedge u_1^* \otimes u_2^* + u_2^* \wedge v^* \otimes u_1^*$, hence,

$$A_3(\tilde{T})_{XYZ} = \underset{XYZ}{\mathfrak{S}} \tilde{T}_{XYZ}.$$

Therefore, $\tilde{\mathcal{T}}_2(V)$ is generated by $\tilde{T}_{XYZ} = S_{YZX} - S_{XZY}$ with $S \in S^2V^* \otimes V^*$ and $\mathrm{s}_{13}(S) = 0$. The first condition is equivalent to $\tilde{T} \in \ker(A_3)$, or equivalently, $\underset{XYZ}{\mathfrak{S}} \tilde{T}_{XYZ} = 0$. The second condition is equivalent $\mathrm{t}_{12}(T) = 0$ straightforwardly. For the explicit expressions of $\tilde{\mathcal{T}}_4(V)$, given

$$\tilde{T} = \frac{1}{2} \sum_{i,j,k=1}^{2n} \tilde{T}_{e_ie_je_k} e_i^* \wedge e_j^* \otimes e_k^* \in \wedge^2 V^* \otimes V^*,$$

we have

$$\begin{aligned}
C(\tilde{T}) &= \frac{1}{2} \sum_{i,j,k=1}^{2n} \tilde{T}_{e_ie_je_k} C(e_i^* \wedge e_j^* \otimes e_k^*) \\
&= \frac{1}{2} \sum_{i,j,k=1}^{2n} \tilde{T}_{e_ie_je_k} \left(\omega_{e_ie_j} e_k^* + \omega_{e_ke_i} e_j^* + \omega_{e_je_k} e_i^* \right) \\
&= \frac{1}{2} \left(\sum_{i,j,k=1}^{2n} \tilde{T}_{e_ie_je_k} \omega_{e_ie_j} e_k^* + \sum_{i,j,k=1}^{2n} \tilde{T}_{e_ie_je_k} \omega_{e_ke_i} e_j^* + \sum_{i,j,k=1}^{2n} \tilde{T}_{e_ie_je_k} \omega_{e_je_k} e_i^* \right).
\end{aligned}$$

By reordering the indices, we group the three summations in only one,

$$C(\tilde{T}) = \frac{1}{2} \sum_{i,j,k=1}^{2n} \left(\tilde{T}_{e_ie_je_k} + \tilde{T}_{e_ke_ie_j} + \tilde{T}_{e_je_ke_i} \right) \omega_{e_ie_j} e_k^*.$$

Thus, we apply that $\{e_1, \dots, e_{2n}\}$ is a symplectic basis,

$$C(\tilde{T}) = \sum_{k=1}^{2n}\sum_{i=1}^{n}\frac{1}{2}\left(\tilde{T}_{e_ie_{i+n}e_k} + \tilde{T}_{e_ke_ie_{i+n}} + \tilde{T}_{e_{i+n}e_ke_i} - \tilde{T}_{e_{i+n}e_ie_k} - \tilde{T}_{e_ke_{i+n}e_i} - \tilde{T}_{e_ie_ke_{i+n}}\right)e_k^*$$
$$= \sum_{k=1}^{2n}\sum_{i=1}^{n}\left(\tilde{T}_{e_ie_{i+n}e_k} + \tilde{T}_{e_ke_ie_{i+n}} + \tilde{T}_{e_{i+n}e_ke_i}\right)e_k^*.$$

Therefore, for $\tilde{T} \in \wedge^3 V^*$, $C(\tilde{T}) = 0$ is equivalent to $\mathrm{t}_{12}(\tilde{T}) = 0$.

Finally, with respect to the explicit expressions of $\tilde{\mathcal{T}}_3(V)$, given $U^* \in V^*$ with dual element $U \in V$,

$$\eta(U^*) = \sum_{i=1}^{n} e_i^* \wedge e_{i+n}^* \wedge U^*$$
$$= \sum_{i=1}^{n}\left(e_i^* \wedge e_{i+n}^* \otimes U^* + U^* \wedge e_i^* \otimes e_{i+n}^* + e_{i+n}^* \wedge U^* \otimes e_i^*\right),$$

evaluating in X, Y, Z, we infer,

$$\eta(U^*)_{XYZ} = \sum_{i=1}^{n} \Big((x_iy_{i+n} - x_{i+n}y_i)\omega_{UZ} + $$
$$+ (\omega_{UX}y_{i+n} - \omega_{UY}x_{i+n})(-z_i) +$$
$$+ (-x_i\omega_{UY} - (-y_i)\omega_{UX})z_{i+n}\Big)$$
$$= \omega_{XY}\omega_{UZ} + \omega_{YZ}\omega_{UX} + \omega_{ZX}\omega_{UY}$$

Therefore, $\tilde{\mathcal{T}}_3(V)$ has the claimed form. □

Remark 5.2.3. We have the following sums

- $\tilde{\mathcal{T}}_1(V) + \tilde{\mathcal{T}}_2(V) = \{\tilde{T} \in \wedge^2 V^* \otimes V^* : \underset{XYZ}{\mathfrak{S}} T_{XYZ} = 0\}$,
- $\tilde{\mathcal{T}}_2(V) + \tilde{\mathcal{T}}_4(V) + W = \{\tilde{T} \in \wedge^2 V^* \otimes V^* : \mathrm{t}_{12}(\tilde{T}) = 0\}$,
- $\tilde{\mathcal{T}}_3(V) + \tilde{\mathcal{T}}_4(V) = \wedge^3 V^*$,
- $\tilde{\mathcal{T}}_2(V) + \tilde{\mathcal{T}}_4(V) = \{\tilde{T} \in \wedge^2 V^* \otimes V^* : \mathrm{t}_{12}(\tilde{T}) = 0, \mathrm{t}_{13}(\tilde{T}) = 0\}$.

Where,

$$W = \left\{\tilde{T} \in \wedge^2 V^* \otimes V^* : \tilde{T}_{XYZ} = \omega_{XY}\omega_{ZU} - n\omega_{XZ}\omega_{YU} + n\omega_{YZ}\omega_{XU}, U \in V\right\}$$

and

$$\mathrm{t}_{13}(\tilde{T})(Y) = \sum_{i=1}^{n} \left(\tilde{T}_{e_i Y e_{i+n}} - \tilde{T}_{e_{i+n} Y e_i} \right)$$

The first two come directly from the expressions of the classes in the previous Theorem and the fact that W is the linear subspace of $\tilde{\mathcal{T}}_2(V) + \tilde{\mathcal{T}}_3(V)$ whose elements vanish for t_{12}. With respect to the third identity, we note that $\mathrm{Id}_{V^*} = \frac{1}{3(n-1)} C \circ \eta$, so that we can decompose $\wedge^3 V^* = \ker C + \operatorname{Im} \eta = \tilde{\mathcal{T}}_3(V) + \tilde{\mathcal{T}}_4(V)$. The last identity is a consequence of the fact that t_{13} vanishes on $\tilde{\mathcal{T}}_2(V) + \tilde{\mathcal{T}}_4(V)$ and does not vanish on W.

5.3 Classifications for almost symplectic and Fedosov AS-manifolds

5.3.1 Almost symplectic AS-manifolds

We now want to classify the infinitesimal models in the case of vector spaces V endowed with a linear symplectic tensor $K = \omega$. If $(V, \tilde{R}, \tilde{T})$ and $(V', \tilde{R}', \tilde{T}')$ are two infinitesimal models associated with symplectic linear tensors ω and ω', respectively, with $\dim V = \dim V'$, since there are symplectomorphisms between V and V', we can identify V' with V and ω' with ω. From (4.10), isomorphisms $f : V \longrightarrow V$ of almost symplectic infinitesimal models satisfy

$$f\tilde{R} = \tilde{R}', \quad f\tilde{T} = \tilde{T}', \quad f\omega = \omega,$$

and in particular $f \in \mathrm{Sp}(V, \omega) = \mathrm{Sp}(V)$. If we decompose curvature-like or torsion-like tensor spaces in $\mathrm{Sp}(V)$-irreducible submodules, then we get a necessary condition to be isomorphic as models, by virtue of Thm. 4.3.3, also as AS-manifolds.

For the classification of the torsion $\tilde{T}$ into $\mathrm{Sp}(V)$-classes, we will work both with $(1,2)$-tensors and $(0,3)$-tensors given by the isomorphism

$$\tilde{T}_{XYZ} = \omega\left(\tilde{T}_X Y, Z\right), \quad X, Y, Z \in V.$$

Let (M, ω) be an almost symplectic AS-manifold. We denote by $\tilde{\mathcal{T}}$ the set of *homogeneous almost symplectic torsions*, that is, the torsions of an AS-connection on (M, ω). Given any $p_0 \in M$, from Thm. 4.3.1, $(V = T_{p_0}M, \tilde{R}_{p_0}, \tilde{T}_{p_0})$ is an infinitesimal model associated with ω_{p_0}. Thus $T_{p_0} \in \wedge^2 V \otimes V$, and the classification given in Thm. 5.2.2 gives us the following decomposition

$$\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_1 + \tilde{\mathcal{T}}_2 + \tilde{\mathcal{T}}_3 + \tilde{\mathcal{T}}_4$$

where

$$\begin{aligned}
\tilde{\mathcal{T}}_1 &= \left\{\tilde{T} \in \tilde{\mathcal{T}} : \tilde{T}_{XYZ} = 2\omega_{XY}\omega_{ZU} + \omega_{XZ}\omega_{YU} - \omega_{YZ}\omega_{XU}, U \in \mathfrak{X}(M)\right\},\\
\tilde{\mathcal{T}}_2 &= \left\{\tilde{T} \in \tilde{\mathcal{T}} : \underset{XYZ}{\mathfrak{S}} \tilde{T}_{XYZ} = 0, \mathrm{t}_{12}(\tilde{T}) = 0\right\}\\
\tilde{\mathcal{T}}_3 &= \left\{\tilde{T} \in \tilde{\mathcal{T}} : \tilde{T}_{XYZ} = \omega_{XY}\omega_{UZ} + \omega_{YZ}\omega_{UX} + \omega_{ZX}\omega_{UY}, U \in \mathfrak{X}(M)\right\},\\
\tilde{\mathcal{T}}_4 &= \left\{\tilde{T} \in \tilde{\mathcal{T}} : \tilde{T}_{XYZ} = -\tilde{T}_{XZY}, \mathrm{t}_{12}(\tilde{T}) = 0\right\}.
\end{aligned}$$

Definition 5.3.1. Let $\tilde{T} \in \tilde{\mathcal{T}}$, $T \neq 0$ be a homogeneous almost symplectic torsion. It is said to be of *type* i (resp. type $i+j$ or $i+j+k$) if $\tilde{T}$ lies in $\tilde{\mathcal{T}}_i$ (resp. $\mathcal{T}_i + \mathcal{T}_j$ or $\mathcal{T}_i + \mathcal{T}_j + \mathcal{T}_k$).

Almost symplectic AS-manifolds may thus belong to sixteen classes defined by its torsion tensor.

Theorem 5.3.2. *Let (M,ω) be an almost symplectic AS-manifold. Then, (M,ω) is a symplectic manifold if and only if the torsion of $\tilde{\nabla}$ lies in $\tilde{\mathcal{T}}_1 + \tilde{\mathcal{T}}_2$.*

Proof. If (M,ω) is a symplectic manifold, there is a torsion-free symplectic connection ∇ (see [AP15, Thm. 2.1]). The difference $S = \nabla - \tilde{\nabla}$ is a $(1,2)$-tensor such that $\tilde{T}_X Y = S_Y X - S_X Y$. Then $\tilde{T}_{XYZ} = A_2(-S) = S_{XZY} - S_{YZX}$, where $S_{XYZ} = \omega(S_Z X, Y)$ and $\tilde{T}_{XYZ} = \omega(\tilde{T}_X Y, Z)$. In particular, $\tilde{T}$ lies in $\mathcal{T}_1 + \mathcal{T}_2$.

Conversely, if $\tilde{T}$ lies in $\mathcal{T}_1 + \mathcal{T}_2$, then there exists at least one tensor $S \in S^2T^*M \otimes T^*M$, such that, $\tilde{T}_{XYZ} = S_{YZX} - S_{XZY}$. We can consider the tensor $S_X Y$ with $\omega(S_Z X, Y) = S_{XYZ}$. It follows that $\tilde{T}_X Y = S_X Y - S_Y X$ with $\omega(\tilde{T}_X Y, Z) = \tilde{T}_{XYZ}$ and S preserves the symplectic form. The connection $\nabla = \tilde{\nabla} - S$ is symplectic. □

5.3.2 Fedosov AS-manifolds

We now want to study infinitesimal models associated with a linear symplectic tensor ω and a homogeneous structure S as in Cor. 4.3.8. Let $(V, \tilde{R}, \tilde{T})$ and $(V', \tilde{R}', \tilde{T}')$ be two infinitesimal models associated with (1,2) linear tensors S and S', respectively, with

$$\begin{aligned}
T_X Y &= \tilde{T}_X Y + S_X Y - S_Y X, \quad R_{XY} = \tilde{R}_{XY} + [S_Y, S_X] - S_{\tilde{T}_X Y},\\
T'_X Y &= \tilde{T}'_X Y + S'_X Y - S'_Y X, \quad R'_{XY} = \tilde{R}'_{XY} + [S'_Y, S'_X] - S'_{\tilde{T}'_X Y},
\end{aligned}$$

and also associated with symplectic linear tensors ω and ω', respectively, with $\dim V = \dim V'$. Since there are symplectomorphisms between V and V', we can identify V with V' and ω with

ω'. Therefore, by (4.14), there is a linear isomorphism $f\colon V \longrightarrow V$ such that,

$$fR = R', \quad fT = T', \quad fS = S', \quad f\omega = \omega, \tag{5.2}$$

and in particular $f \in \mathrm{Sp}(V,\omega) = \mathrm{Sp}(V)$. If we decompose cotorsion-like, curvature-like or torsion-like tensor spaces in $\mathrm{Sp}(V)$-irreducible submodules, then we get a necessary condition to be isomorphic as models, by virtue of Thm. 4.3.3, also as AS-manifolds.

Let (M,ω,∇) be a Fedosov manifold ($\dim M = 2n$), that is, a symplectic manifold equipped with affine and torsion free connection such that $\nabla\omega = 0$ (cf. [GRS98]). Let $S = \nabla - \tilde{\nabla}$ be a homogeneous structure tensor, i. e.,

$$\tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}\omega = 0.$$

Since $\nabla\omega = 0$, the second condition is equivalent to $S\cdot\omega = 0$. We will work with S both as a $(1,2)$-tensor and a $(0,3)$-tensor by the isomorphism

$$S_{XYZ} = \omega(S_Z X, Y).$$

The condition $S\cdot\omega = 0$ is equivalent to

$$S_{XYZ} = S_{YXZ}$$

that is, $S \in S^2T^*M \otimes T^*M$.

From Thm. 4.3.1, given $p_0 \in M$, we can consider the infinitesimal model $(V = T_{p_0}M, \tilde{R}_{p_0}, \tilde{T}_{p_0})$ associated with S_{p_0} and ω_{p_0} with

$$\begin{aligned}
(T_{p_0})_X Y &= (\tilde{T}_{p_0})_X Y + (S_{p_0})_X Y - (S_{p_0})_Y X,\\
(R_{p_0})_{XY} &= (\tilde{R}_{p_0})_{XY} + [(S_{p_0})_Y, (S_{p_0})_X] - (S_{p_0})_{(\tilde{T}_{p_0})_X Y},
\end{aligned}$$

where R_{p_0} and T_{p_0} are the curvature and torsion of ∇ at p_0 and $S_{p_0} \in S^2V^* \otimes V^*$. We denote by $\mathcal{S}$ the set of homogeneous structures on a Fedosov manifold (M,ω,∇). Hence, by Thm. 5.2.1, we have the following classification of homogeneous structure tensors in $\mathrm{Sp}(V)$-invariant subspaces:

$$\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3,$$

where

$$\begin{aligned}
\mathcal{S}_1 &= \Big\{ S \in \mathcal{S} : S_{XYZ} = \omega_{ZY}\omega_{XU} + \omega_{ZX}\omega_{YU},\, U \in \mathfrak{X}(M) \Big\}, \\
\mathcal{S}_2 &= \Big\{ S \in \mathcal{S} : \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0,\, \mathrm{s}_{13}(S) = 0 \Big\}, \\
\mathcal{S}_3 &= \Big\{ S \in \mathcal{S} : S_{XYZ} = S_{XZY} \Big\},
\end{aligned}$$

and

$$\mathrm{s}_{13}(S)(Z) = \sum_{i=1}^{n} \left(S_{e_i Z e_{i+n}} - S_{e_{i+n} Z e_i} \right),$$

for a symplectic basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ of $T_{p_0}M$.

Definition 5.3.3. Let $S \in \mathcal{S}$ $S \neq 0$, be a homogeneous Fedosov structure. It is of *type i* if S lies in $\mathcal{S}_i$ and correspondingly it is of *type* $i+j$ if S lies in $\mathcal{S}_i + \mathcal{S}_j$ with $i, j \in \{1,2,3\}$ and $i \neq j$.

Hence, Fedosov homogeneous structure are classified into eight different classes.

Remark 5.3.4. In [Vai85] the author gives a decomposition of the curvature tensor of a symplectic connection in two $\mathrm{Sp}(V)$-irreducible submodules: Ricci type and Ricci flat. Hence, by virtue of (5.2) and Thm. 4.3.3, there are four different classes of symplectic curvature tensor of Fedosov AS-manifolds. We can combine this idea to refine the classification in Def. 5.3.3 to get as many as thirty-two different classes of Fedosov AS-manifolds.

With respect to the classification of homogeneous structures in Def. 5.3.3 and the classification of torsions $\tilde{T}$ o AS-manifolds in Def. 5.3.1, we have the following result which is a consequence of the expression of A_2 in (5.1).

Proposition 5.3.5. *Let (M, ω, ∇) be a Fedosov manifold equipped with homogeneous structure S.*

- *If $S \in \mathcal{S}_1$, then the torsion $\tilde{T}$ of $\tilde{\nabla} = \nabla - S$ belongs to $\tilde{\mathcal{T}}_1$.*
- *If $S \in \mathcal{S}_2$, then the torsion $\tilde{T}$ of $\tilde{\nabla} = \nabla - S$ belongs to $\tilde{\mathcal{T}}_2$.*
- *If $S \in \mathcal{S}_3$, then the torsion $\tilde{T}$ of $\tilde{\nabla} = \nabla - S$ vanishes. The manifold $(M, \omega, \tilde{\nabla})$ is a Fedosov manifold with parallel curvature.*

5.4 Fedosov AS-manifold of linear type.

Let (M, ω, ∇) be a Fedosov manifold equipped with an AS-homogeneous structure S, that is

$$\tilde{\nabla} R = 0, \quad \tilde{\nabla}\omega = 0, \quad \tilde{\nabla} S = 0,$$

for $S = \nabla - \tilde{\nabla}$.

Definition 5.4.1. A homogeneous Fedosov structure S in (M, ω, ∇) is said to be of *linear type* if it belongs to the class $\mathcal{S}_1$, that is

$$S_X Y = \omega(X,Y)\xi - \omega(Y,\xi)X, \tag{5.3}$$

for a vector field $\xi \in \mathfrak{X}(M)$.

As we noticed in the metric setting, these are called of linear type because of the dimension of the class $\mathcal{S}_1$ grows linearly with the dimension of the manifold.

Theorem 5.4.2. *A Fedosov manifold (M, ω, ∇) admitting a homogeneous structure tensor of linear type does not admit any pseudo-Riemannian metric such that $S \cdot g = 0$.*

Proof. Let η be a vector such that $\omega(\eta, \xi) = 1$. Let g be a pseudo-Riemannian metric such that $S \cdot g = 0$, that is

$$\begin{aligned} 0 &= g(S_X Y, Z) + g(Y, S_X Z) \\ &= \omega(X,Y)g(\xi,Z) - \omega(Y,\xi)g(X,Z) + g(Y,\xi)\omega(X,Z) - g(Y,X)\omega(Z,\xi). \end{aligned}$$

Taking $Y = Z = \eta$ we get $g(X,\eta) = g(\eta,\xi)\omega(X,\eta)$. We then get $g(\eta,X) = 0$ for $X = \xi$ and for $X \in \{v : \omega(v,\xi) = 0\}$, which is impossible since g is not degenerate. □

Remark 5.4.3. This last theorem shows that homogeneous structure tensors on Fedosov manifolds can never be studied under the perspective of Kiričenko's Theorem as they are genuine non-metric homogeneous objects.

We fix the notation of curvature and torsion tensor fields associated with one connection ∇,

$$\begin{aligned} R_{XY}Z &= \nabla_{[X,Y]}Z - \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z), \\ T_X Y &= \nabla_X Y - \nabla_Y X - [X,Y], \end{aligned}$$

for the curvature, we will work both with $(1,3)$-tensors and $(0,4)$-tensors given by the isomorphism,

$$R_{XYZU} = \omega(R_{XY}Z, U), \quad X, Y, Z, U \in TM.$$

Proposition 5.4.4. *The condition $\tilde{\nabla} S = 0$ is equivalent to $\tilde{\nabla}\xi = 0$.*

Proof. Substituting (5.3) in $(\tilde{\nabla}_X S)(Y,Z) = \tilde{\nabla}_X (S_Y Z) - S_{\tilde{\nabla}_X Y} Z - S_Y(\tilde{\nabla}_X Z)$, we get

$$\begin{aligned}(\tilde{\nabla}_X S)(Y,Z) &= \tilde{\nabla}_X \left(\omega(Y,Z)\xi - \omega(Z,\xi)Y\right) \\ &\quad - \omega(\tilde{\nabla}_X Y, Z)\xi + \omega(Z,\xi)\tilde{\nabla}_X Y \\ &\quad - \omega(Y,\tilde{\nabla}_X Z)\xi + \omega(\tilde{\nabla}_X Z,\xi)Y \\ &= X\left(\omega(Y,Z)\right)\xi + \omega(Y,Z)\tilde{\nabla}_X \xi - X\left(\omega(Z,\xi)\right)Y - \omega(Z,\xi)\tilde{\nabla}_X Y \\ &\quad - \omega(\tilde{\nabla}_X Y, Z)\xi \qquad\qquad\qquad\qquad + \omega(Z,\xi)\tilde{\nabla}_X Y \\ &\quad - \omega(Y,\tilde{\nabla}_X Z)\xi \qquad\qquad + \omega(\tilde{\nabla}_X Z,\xi)Y.\end{aligned}$$

Using $X(\omega(Y,Z)) = \omega(\tilde{\nabla}_X Y, Z) + \omega(Y, \tilde{\nabla}_X Z)$ (that is $\tilde{\nabla}\omega = 0$), we collect the columns and we get

$$(\tilde{\nabla}_X S)(Y,Z) = \omega(Y,Z)\tilde{\nabla}_X \xi - \omega(Z,\tilde{\nabla}_X \xi)Y.$$

If $\tilde{\nabla}_X S = 0$, then $\tilde{\nabla}_X \xi$ is proportional to any vector field Y, and hence $\tilde{\nabla}_X \xi = 0$. Conversely, if $\tilde{\nabla}_X \xi = 0$, then, we substitute in the previous equation and get $\tilde{\nabla}_X S = 0$. □

In particular, the vector field ξ defining a homogeneous structure satisfies

$$\nabla_X \xi = S_X \xi = \omega(X,\xi)\xi. \tag{5.4}$$

Following the ideas of classifications of homogeneous structures of linear type in the pseudo-Riemannian case (see [CC19, Ch. 5]), we study the curvature and Ricci tensor of ∇.

Proposition 5.4.5. *Fedosov AS-manifolds of linear type satisfy,*

$$R_{\xi XYZ} = \omega_{X\xi}\omega_{Y\xi}\omega_{Z\xi}R_{\xi\xi^\perp\xi^\perp\xi^\perp}, \tag{5.5}$$

$$\begin{aligned}R_{XYUW} = &\left(-\omega_{XY} - \omega_{\xi^\perp X}\omega_{Y\xi} + \omega_{\xi^\perp Y}\omega_{X\xi}\right)\omega_{U\xi}\omega_{W\xi}R_{\xi\xi^\perp\xi^\perp\xi^\perp} \\ &- \omega_{X\xi}R_{Y\xi^\perp UW} - \omega_{Y\xi}R_{\xi^\perp XUW}\end{aligned} \tag{5.6}$$

for every X, Y, U, $W \in TM$ and any $\xi^\perp \in TM$ such that $\omega(\xi^\perp,\xi) = 1$.

Proof. From (5.4) and the fact that the torsion of ∇ vanishes, we get

$$\begin{aligned}R_{XY}\xi &= \omega([X,Y],\xi)\xi - \nabla_X(\omega(Y,\xi)\xi) + \nabla_Y(\omega(X,\xi)\xi) \\ &= \omega(\nabla_X Y,\xi)\xi - \omega(\nabla_Y X,\xi)\xi - X(\omega(Y,\xi))\xi + Y(\omega(X,\xi))\xi.\end{aligned}$$

As $\nabla\omega = 0$, this last expression simplifies to $R_{XY}\xi = -\omega(Y, \nabla_X\xi)\xi + \omega(X, \nabla_Y\xi)\xi$ which, again by (5.4), gives

$$R_{XY}\xi = 0, \tag{5.7}$$

together with

$$R_{\xi X}Y = R_{\xi Y}X, \tag{5.8}$$

from the first Bianchi identity.

As $R_{XY}\cdot\omega = 0$, then,

$$R_{XYZU} = R_{XYUZ}. \tag{5.9}$$

The condition $\tilde{\nabla}R = 0$ (that is, $\nabla_X R = S_X R$) reads

$$(\nabla_X R)_{YZUW} = -R_{S_XY\,ZUW} - R_{Y\,S_XZUW} - R_{YZS_XUW} - R_{YZU\,S_XW}.$$

Applying the second Bianchi identity, we get

$$\begin{aligned} 0 =& \underset{XYZ}{\mathfrak{S}} \left(-R_{S_XY\,ZUW} - R_{Y\,S_XZUW} - R_{YZS_XUW} - R_{YZU\,S_XW}\right) \\ =& \underset{XYZ}{\mathfrak{S}} \big(-\omega_{XY}R_{\xi ZUW} - \omega_{XZ}R_{Y\xi UW} - \omega_{XU}R_{YZ\xi W} - \omega_{XW}R_{YZU\xi} \\ &+\omega_{Y\xi}R_{XZUW} + \omega_{Z\xi}R_{YXUW} + \omega_{U\xi}R_{YZXW} + \omega_{W\xi}R_{YZUX}\big), \end{aligned}$$

which by virtue of (5.7), (5.9) and the first Bianchi identity reduces to

$$0 = \underset{XYZ}{\mathfrak{S}} \left(\omega_{XY}R_{\xi ZUW} + \omega_{X\xi}R_{YZUW}\right). \tag{5.10}$$

Choosing $Z = \xi$, we have

$$\omega_{X\xi}R_{\xi YUW} = \omega_{Y\xi}R_{\xi XUW}. \tag{5.11}$$

If we choose $X = \xi^\perp$ in (5.11), then, we get $R_{\xi YUW} = \omega_{Y\xi}R_{\xi\xi^\perp UW}$, using symmetry of (5.9) and applying equation above we have $R_{\xi YUW} = \omega_{Y\xi}\omega_{U\xi}R_{\xi\xi^\perp\xi^\perp W}$, and proceeding in an analogous way, using (5.8) and (5.9), we conclude (5.5). Substituting $Z = \xi^\perp$ in (5.10) and using (5.5), we get (5.6). $\square$

Remark 5.4.6. Equation (5.10) can be refined using (5.5),

$$0 = \underset{XYZ}{\mathfrak{S}} \left(\omega_{XY}\omega_{Z\xi}\omega_{U\xi}\omega_{W\xi}R_{\xi\xi^\perp\xi^\perp\xi^\perp} + \omega_{X\xi}R_{YZUW}\right).$$

We now proceed by parts to prove the main result of this section (see Thm. 5.4.9) which characterize Fedosov AS-manifolds of linear type in terms of a foliation of Hamiltonian leaves of codimension 1.

Lemma 5.4.7. *The distribution* $\mathcal{D} = \{X \in \mathfrak{X}(M) : \omega(X, \xi) = 0\}$ *is an integrable distribution.*

Proof. Given $X, Y \in \mathcal{D}$, we have

$$\omega([X,Y],\xi) = \omega(\nabla_X Y, \xi) - \omega(\nabla_Y X, \xi).$$

As $\nabla\omega = 0$, then

$$\omega([X,Y],\xi) = X(\omega(Y,\xi)) - \omega(Y, \nabla_X \xi) - Y(\omega(X,\xi)) + \omega(X, \nabla_Y \xi).$$

Finally, because of (5.4),

$$\omega([X,Y],\xi) = -\omega(X,\xi)\omega(Y,\xi) + \omega(Y,\xi)\omega(X,\xi) = 0.$$

Hence, the distribution $\mathcal{D}$ is integrable. □

Lemma 5.4.8. *The vector field* ξ *satisfies the following properties,*

1. *It is a geodesic vector field with respect to* ∇.
2. *Its flow preserves the symplectic form.*

Proof. The first statement comes from (5.4), that is, $\nabla_\xi \xi = 0$. With respect to the second item, using $\nabla\omega = 0$ and that ∇ is torsion free, for X, Y two vector fields

$$\begin{aligned}(\mathcal{L}_\xi \omega)(X,Y) &= \omega(\nabla_\xi X, Y) - \omega(\nabla_\xi X, Y) - \omega(X, \nabla_\xi Y)\\ &\quad + \omega(X, \nabla_\xi Y) + \omega(\nabla_X \xi, Y) + \omega(X, \nabla_Y \xi)\\ &= \omega(X,\xi)\omega(\xi,Y) + \omega(X,\xi)\omega(Y,\xi) = 0,\end{aligned}$$

which means that the flow of ξ preserves the symplectic form. □

Theorem 5.4.9. *Let* (M, ω, ∇) *be a connected and simply-connected Fedosov manifold endowed with a homogeneous structure* S *of linear type. Let* ξ *be the vector field associated with* S*. Then:*

- *the manifold is foliated by the leaves* $H = c_0$, $c_0 \in \mathbb{R}$, *where* H *is the Hamiltonian defined by the Hamilton equation* $i_\xi \omega = dH$,
- *the connection* ∇ *restricts to the leaves and in particular, the leaves are totally geodesic submanifolds,*
- *the leaves are flat manifolds.*

Furthermore, if in addition $\tilde{\nabla} = \nabla - S$ is complete, M is Fedosov homogeneous.

Proof. As M is simply-connected, the invariance of ω with respect to ξ gives the existence of a function $H : M \longrightarrow \mathbb{R}$ solving the Hamilton equation $i_\xi \omega = dH$. For any $X \in \mathcal{D}$, we have $dH(X) = \omega(\xi, X) = 0$, so that H is constant along the leaves of $\mathcal{D}$. For X, Y vector fields tangent to the distribution, $\omega(\nabla_X Y, \xi) = X(\omega(\xi, Y)) + \omega(\nabla_X \xi, Y) = 0$, and the connection restricts to the leaves. Finally, the curvature of ∇ vanishes along the leaves by (5.6). □

The local version of this last result is straightforward.

Theorem 5.4.10. *Let (M, ω, ∇) be a Fedosov manifold endowed with a homogeneous structure S of linear type. Let ξ be the vector field associated with S. Then:*

- *the manifold is foliated by the leaves locally defined by a Hamiltonian H of the Hamilton equation $i_\xi \omega = dH$,*
- *the connection ∇ restricts to the leaves and in particular, the leaves are totally geodesic submanifolds,*
- *the leaves are flat manifolds.*

Furthermore, M is locally Fedosov homogeneous.

We finish with two examples of Fedosov homogeneous structures of linear type.

Example 5.4.11. Let $M = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ be the half-plane endowed with a Fedosov structure defined by the symplectic form $\omega = \frac{1}{3x^2} \mathrm{d}x \wedge \mathrm{d}y$ and the connection ∇ given by the following non-vanishing Christoffel symbols

$$\Gamma^1_{11} = -\frac{4}{3x}, \qquad \Gamma^2_{12} = \frac{2}{3x}, \qquad \Gamma^2_{21} = -\frac{2}{3x}.$$

We consider the vector field

$$\xi = x\frac{\partial}{\partial y},$$

and the tensor field

$$S_X Y = \omega_{XY}\xi - \omega_{Y\xi}X.$$

One can check that

$$\nabla\omega = 0, \quad T = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0,$$

where R is the curvature of ∇ and $\tilde{\nabla} = \nabla - S$, that is, S is a homogeneous structure of linear type as in Thm. 5.4.10. The manifold M is foliated by the leaves $(\{x = \text{constant}\})$ defined by the Hamiltonian $H(x,y) = \frac{-1}{3}\log(x)$, the connection ∇ restricts to the leaves, and they are totally geodesic and flat. Furthermore, since the vector fields $x\partial/\partial x$ and $\xi = x\partial/\partial y$ are complete geodesic vector fields of $\tilde{\nabla}$, this connection is complete and M is homogeneous Fedosov manifold as in Thm. 5.4.9. With respect to the group acting transitively on M, it turns out that $\tilde{R} = 0$. The Nomizu construction (4.11) gives $M = G$, with $G = \mathrm{Aff}(1)_0$ the connected component of the identity of the group of affine transformation of $\mathbb{R}$.

If, instead, we consider

$$\Gamma_{11}^{1} = -\frac{2}{x},$$

as the only non-vanishing Christoffel symbol, and (for the sake of convenience with the computations) $\omega = \frac{1}{x^2}\mathrm{d}x \wedge \mathrm{d}y$, we again get that (M, ω, ∇) is a homogeneous Fedosov manifold of linear type as in Thm. 5.4.9 with $\xi = x\partial/\partial y$. In this case, one can check that

$$\tilde{R}_{\xi\eta}\eta = -2\xi, \qquad \tilde{R}_{\xi\eta}\xi = 0,$$

with $\eta = x\partial/\partial x + y\partial/\partial y$. The algebra $\mathfrak{g} = \mathrm{span}\{\xi, \eta, \tilde{R}_{\xi\eta}\}$ built from the Nomizu construction is the Lie algebra of the three dimensional Lie group acting on M such that $M = G/H$ with $H \simeq \mathbb{R}$. One checks that, following the convention of [Bia01, p. 2194], the Lie algebra $\mathfrak{g}$ of G is the Lie algebra of type Bianchi VI, with real parameter $h = 2$.

Chapter 6

The Ambrose-Singer Theorem for cohomogeneity one Riemannian manifolds

The Tricerri-Vanhecke program was exclusive for pseudo-Riemannian geometry. However, in Ch. 4 and Ch. 5, we learnt that these techniques can be applied to non-metric frameworks, such as symplectic homogeneous manifolds. This opens the door to thinking about new generalizations of the Ambrose-Singer Theorem for non-transitive actions.

A cohomogeneity one action on a Riemannian manifold is an isometric action whose principal orbits are hypersurfaces. Thus, the action is non-transitive. If a group acts on a Riemannian manifold with cohomogeneity one, we say that such a manifold is a cohomogeneity one manifold. These are precisely the objects of study in this chapter. Techniques based on cohomogeneity one actions have successfully been used to find examples of interesting geometric structures, such as manifolds with positive curvature, Einstein metrics, or Ricci solitons.

The aim of this chapter is to initiate a line of research, analogous to Tricerri-Vanhecke's program, adapted to cohomogeneity one Riemannian manifolds. This objective is pursued by achieving the following goals.

In Sec. 6.2, we characterize connected, simply-connected, and complete cohomogeneity one Riemannian manifolds by the existence of a geodesically complete linear connection satisfying certain geometric covariant derivative equations. This connection is called cohomogeneity one AS-connection. Analogous to the transitive framework, we introduce the difference tensor $S = \nabla - \tilde{\nabla}$ where $\tilde{\nabla}$ is the connection under consideration and ∇ is the Levi-Civita connection. This is called the cohomogeneity one structure. Afterwards, in Sec. 6.3, we relax the topological conditions and characterize locally cohomogeneity one Riemannian manifolds, that is, we assume that there is a Lie pseudo-group of isometries acting on the manifold whose principal orbits are hypersurfaces. The existence of cohomogeneity one structures describes the nature of being a locally cohomogeneity one manifold.

Following the program of [TV83] or [CC19], but with a non-metric perspective, i. e., we pursue the line of work developed in Ch. 5. Sec. 6.4 decomposes the space of the tensor elements of cohomogeneity one structures in $\mathrm{SO}(n)$-irreducible submodules. This decomposition yields four submodules for cohomogeneity structures. One is the homogeneous structure in the leaf; therefore, we can scrutinize the cohomogeneity one Riemannian action by examining the projection onto this submodule. Another is the second fundamental form of the leaves.

In Sec. 6.5, we give a formula for the cohomogeneity one structure constructed in the main theorem of Sec. 6.2. Afterwards, we study the uniqueness of this cohomogeneity one structure depending on the action. Finally, Sec. 6.6 is dedicated to giving examples of cohomogeneity one structures in the Euclidean space and the real hyperbolic space.

6.1 Cohomogeneity one Riemannian manifolds

We now give a brief introduction to cohomogeneity one Riemannian manifolds. We refer the reader to [BCO16] for an introduction to cohomogeneity one actions.

Let (M,g) be a Riemannian manifold and let G be a Lie subgroup of the Lie group $\mathrm{Isom}(M,g)$ of isometries. The action of G may be non-transitive. According to the notation in [AA93], we say that (M,g) is a Riemannian G-manifold. The *orbit space* of p is $G\cdot p = \{f\cdot p \in M :\ f\in G\}$ and the *isotropy group* at p is $G_p = \{f\in G :\ f\cdot p = p\}$. The orbit space is an immersed submanifold and the isotropy group is a subgroup of G. In particular, if G acts transitively, so $G\cdot p = M$, then (M,g) is a Riemannian homogeneous manifold. Until now, we have studied transitive actions. From now on, we assume that the action is non-transitive.

Let (M,g) be a Riemannian G-manifold and let (M',g') be another Riemannian G'-manifold. These two are *isomorphic* if there exists an isometry $F\colon (M,g)\longrightarrow (M',g')$ and a Lie group homomorphism $\gamma\colon G\longrightarrow G'$ such that $F(f\cdot p) = \gamma(f)\cdot F(p)$.

The *space of orbits* of the G-manifold is the quotient $M/G := M/\sim$ with respect to the relation $p\sim q$ if and only if $q = G\cdot p$. For example, we consider the flat torus $(\mathbb{T}^2,g)$ with the $G=\mathbb{R}$ action given by

$$\begin{aligned} \mathbb{R}\times\mathbb{T}^2 &\longrightarrow \mathbb{T}^2\\ (t,[x,y]) &\longmapsto [x+t, y+\lambda t], \quad \lambda\in\mathbb{R}. \end{aligned}$$

For this action, if $\lambda\in\mathbb{Q}$, then M/G is homeomorphic to $\mathbb{S}^1$. Otherwise, if $\lambda\in\mathbb{R}-\mathbb{Q}$, then M/G is not even Hausdorff. In this case, the orbits are immersed non embedded submanifolds. In general, this type of orbits makes the study of the action extremely difficult.

This obstruction is overcome by assuming the isometric action to be proper. An isometric action G on (M,g) is *proper* if and only if

$$\begin{aligned} G\times M &\longrightarrow M\times M \\ (f,p) &\longmapsto (p,f\cdot p) \end{aligned}$$

satisfies that the preimage of any compact set of $M\times M$ is compact in $G\times M$. For proper actions, the space of orbits is Hausdorff and each orbit is an embedded, closed and compact submanifold. From now on, assume that M/G is connected and each orbit is an embedded and closed submanifold.

Definition 6.1.1 ([Pal61, Def. 2.1.1])**.** Let (M,g) be a Riemannian G-manifold and H is a closed subgroup of G. A submanifold Σ of M is a *slice* of $p\in M$ if:

1. $p\in\Sigma$.
2. $G\cdot\Sigma$ is an open subset of M.
3. There exists an G-equivariant map $F:G\cdot\Sigma\longrightarrow G/H$ such that $F^{-1}(eH)=\Sigma$, where e is the neutral element of G.

In particular, if the action is proper then there exists a slice at each point, see [Pal61, Thm. 2.3.3]. The main consequence of the existence of slices is that we can define a partial ordering in some classes of the space of orbits. Two orbits $G\cdot p$ and $G\cdot q$ are of the same type if G_p and G_q are conjugate. This is an equivalence relation and the class $[G\cdot p]$ is called the *orbit type* at p. We say that two classes are ordered as $[G\cdot p]\leq[G\cdot q]$ if and only if there is a Lie subgroup in G_p conjugate to G_q. There is an unique largest type of orbits and these orbits are called *principal orbits*. We have that:

- The dimension of each principal orbit is maximal;
- The union of principal orbits is an open and dense subset in M.

If an orbit is of maximal dimension, but non-principal, then it is *exceptional*. If an orbit has lower dimension that principal orbits, then it is *singular*.

Definition 6.1.2. A Riemannian G-manifold (M,g) is a cohomogeneity one Riemannian manifold if the principal orbits are hypersurfaces.

6.2 Characterization of cohomogeneity one manifolds: global version

A Riemannian manifold (M,g) is said to be a regular cohomogeneity one Riemannian manifold if there is a group of isometries $G \subset \mathrm{Isom}(M,g)$ such that M/G is connected and every orbit is orientable, embedded, closed and principal of codimension one. If the action of a subgroup $G \subset \mathrm{Isom}(M,g)$ is proper, and the codimension of a generic orbit is one, then the subset of principal orbits is a regular cohomogeneity one Riemannian manifold.

Lemma 6.2.1. *Let (M,g) be a connected Riemannian manifold equipped with a vector field $\xi \neq 0$. Let $\tilde{\nabla}$ be an affine connection such that,*

$$\tilde{\nabla}\xi = 0, \quad \tilde{\nabla}_X g = 0, \quad \tilde{T}(X,Y) \in \mathcal{D}, \quad \forall X,Y \in \mathcal{D},$$

where $\mathcal{D} = \{X : g(X,\xi) = 0\}$ and $\tilde{T}$ is the torsion of $\tilde{\nabla}$. Then, the distribution $\mathcal{D}$ is integrable and the leaves are totally geodesic submanifolds with respect to $\tilde{\nabla}$.

Proof. We take two vector fields X, Y such that $g(X,\xi) = 0$ and $g(Y,\xi) = 0$. Taking derivatives of these expressions, and using $\tilde{\nabla}_X g = 0$ and $\tilde{\nabla}\xi = 0$, we have that

$$0 = g(\tilde{\nabla}_Y X,\xi) + g(X,\tilde{\nabla}_Y \xi) = g(\tilde{\nabla}_Y X,\xi), \qquad 0 = g(\tilde{\nabla}_X Y,\xi) + g(Y,\tilde{\nabla}_X \xi) = g(\tilde{\nabla}_X Y,\xi).$$

Then $g([X,Y],\xi) = g(-\tilde{T}(X,Y),\xi) = 0$ so that $\mathcal{D}$ is integrable. From the Frobenious Theorem, we have a foliation whose leaves are orthogonal to the vector field ξ. □

Theorem 6.2.2. *Let (M,g) be a connected, simply-connected, orientable and complete Riemannian manifold. Then the following two are equivalent:*

(1) *(M,g) is a regular cohomogeneity one Riemannian manifold.*

(2) *There exists a complete linear connection $\tilde{\nabla}$ and a vector field ξ with $g(\xi,\xi) = 1$, such that,*

$$\begin{aligned} \tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}\xi = 0, \\ \tilde{\nabla}_X g = 0, \quad \tilde{T}(X,Y) \in \mathcal{D}, \quad \forall X,Y \in \mathcal{D}, \end{aligned} \tag{6.1}$$

where $\mathcal{D} = \{X : g(X,\xi) = 0\}$ and $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion of $\tilde{\nabla}$. Furthermore, the maximal integral leaves of the distribution $\mathcal{D}$ (it is integrable according to Lem. 6.2.1) are embedded and closed.

Proof. **The proof of (1) implies (2).** Let G be a Lie group of isometries of M acting on M in such a way that the orbits define a cohomogeneity one foliation.

Since M is orientable, any orbit admits a unit normal vector field ξ.

Lemma 6.2.3. *For any $p \in M$, the geodesic $\exp(t\xi_p)$ intersects all the leaves of the foliation orthogonally.*

Proof. Since M is complete, there is a minimizing geodesic γ starting at p and reaching any orbit O at a point $q \in O$ such that $\gamma(0) = p$ and $\gamma(b) = q$. Let $\mathfrak{g}$ be the Lie algebra of G, for $B \in \mathfrak{g}$, we consider the family of curves $\gamma_\varepsilon(t) = e^{-t\varepsilon B}\gamma(t)$. Then, $\gamma_\varepsilon'(t) = -\varepsilon B^*_{\gamma_\varepsilon(t)} + (L_{e^{-t\varepsilon B}})_*\gamma'(t)$ and

$$g(\gamma_\varepsilon'(t), \gamma_\varepsilon'(t)) = \varepsilon^2 g(B^*_{\gamma_\varepsilon(t)}, B^*_{\gamma_\varepsilon(t)}) - 2\varepsilon g(B^*_{\gamma_\varepsilon(t)}, (L_{e^{-t\varepsilon B}})_*\gamma'(t)) + g(\gamma'(t), \gamma'(t)).$$

On the one hand,

$$g(B^*_{\gamma_\varepsilon(t)}, (L_{e^{-t\varepsilon B}})_*\gamma'(t)) = g(B^*_{\gamma(t)}, \gamma'(t)) = g(B^*_p, \gamma'(0)),$$

where the last identity is due to the fact that

$$\frac{d}{dt} g(B^*_{\gamma(t)}, \gamma'(t)) = g(\nabla_{\gamma'} B^*, \gamma') = 0,$$

since B^* is Killing (and thus, ∇B^* is skew-symmetric). Then the variation of the energy of the geodesic γ is

$$\int_0^b g(\gamma_\varepsilon'(t), \gamma_\varepsilon'(t))dt = \varepsilon^2 \int_0^b g(B^*_{\gamma_\varepsilon(t)}, B^*_{\gamma_\varepsilon(t)})dt - 2b\varepsilon g(B^*_p, \gamma'(0)) + \int_0^b g(\gamma'(t), \gamma'(t))dt,$$

which can be smaller than the energy of $\gamma(t)$ for small ε. This contradicts that γ is minimizing. Thus, $g(B^*_p, \gamma'(0)) = 0$, i. e., $\gamma'(0)$ is orthogonal to $T_p(G \cdot p)$.

Therefore, γ is the geodesic defined by ξ_p. And since we have checked that $g(B^*_{\gamma(t)}, \gamma'(t))$ is constant along γ, we have that γ intersect all the leaves orthogonally. $\square$

Now we consider the differentiable map $\varphi \colon \mathbb{R} \times (G \cdot p) \longrightarrow M$ defined by

$$\varphi_t(g \cdot p) = \exp_{g \cdot p}(t\xi_{g \cdot p}).$$

From the lemma above, this map is surjective. On the other hand, since G sends geodesics to geodesics and $g \cdot \xi_q = \xi_{g \cdot q}$ because G-orbits are principal, we have that

$$g \cdot \varphi_t(q) = \varphi_t(g \cdot q), \qquad \forall q \in G \cdot p.$$

In particular, since all orbits are principal (and thus, all isotropy groups are conjugate),

$$\mathrm{Isot}_G(p) = \mathrm{Isot}_G(\varphi_t(p)),$$

where the isotropy $\mathrm{Isot}_G(q)$ of $q \in M$ is the Lie subgroup $\{h \in G:\ h \cdot q = q\}$.

The vector field ξ initially defined on $G \cdot p$ only, can now be extended to M by simply taking derivatives of $\partial\varphi/\partial t$, $t \in \mathbb{R}$. This is well defined since if we had $q = \varphi_{t_1}(p) = \varphi_{t_2}(g \cdot p)$ with

$$\frac{\partial \varphi_t(p)}{\partial t}(t_1) = -\frac{\partial \varphi_t(g \cdot p)}{\partial t}(t_2), \tag{6.2}$$

then

$$\varphi_{\frac{t_1+t_2}{2}}(p) = \varphi_{\frac{t_1+t_2}{2}}(g \cdot p) = g \cdot \varphi_{\frac{t_1+t_2}{2}}(p),$$

that is, $g \in \mathrm{Isot}_G(\varphi_{\frac{t_1+t_2}{2}}(p)) = \mathrm{Isot}_G(p)$. But the uniqueness of geodesics would make (6.2) impossible for $g \cdot p = p$.

The orbits of ξ are all of the same type. There is thus a group $A = \mathbb{R}$ or $\mathbb{S}^1$ and we have a transitive action on the left

$$\begin{aligned} (A \times G) \times M &\longrightarrow M \\ ((\varphi_t, g), p) &\longmapsto \varphi_t(g \cdot p) = g \cdot \varphi_t(p). \end{aligned}$$

Let $\mathrm{Isot}(p) = \{(\varphi_t, g) \in A \times G:\ \varphi_t(g \cdot p) = p\}$ denote the isotropy of p by the action of $A \times G$. The subgroup $D = \{t \in A:\ \varphi_t(p) \in G \cdot p\} \subset A$ is closed and it is generated by the smallest t_0 such that $\varphi_{t_0}(p) = g \cdot p$. Therefore, $\mathrm{Isot}(p) = \mathcal{L}((\varphi_{t_0}, g^{-1})) + \mathrm{Isot}_G(p)$ where $\mathcal{L}((\varphi_{t_0}, g^{-1}))$ is the subgroup generated by (φ_{t_0}, g^{-1}) in $A \times G$ and $\mathrm{Isot}_G(p)$ is the isotropy group of p by the action of G. As $\mathcal{L}((\varphi_{t_0}, g^{-1}))$ is discrete, the Lie algebra $\mathrm{Isot}(p)$ is the Lie algebra $\mathfrak{h}$ of $\mathrm{Isot}_G(p)$. We have that $[\mathfrak{h}, \mathfrak{a}] = 0$ because G leaves ξ invariant.

Since each orbit is a reductive homogeneous manifold, then we have a connection in every leaf such that the geodesics of the connection are the curves $\exp(tX) \cdot p$ for every $X \in \mathfrak{m}$ and $p \in M$. Moreover, there is a decomposition $\mathfrak{a} + \mathfrak{g} = \mathfrak{h} + (\mathfrak{a} + \mathfrak{m})$ and the subspace $\mathfrak{a} + \mathfrak{m}$ is $\mathrm{Ad}(\mathrm{Isot}_G(p))$-invariant. We have that $\varphi_{t_0} \circ L_{g^{-1}}$ is a global diffeomorphism preserving ξ and the horizontal distribution defined by the connection above, i. e., the subspace $\mathfrak{m}$ is $\mathrm{Ad}(\varphi_{t_0} \circ L_{g^{-1}})$-invariant. It follows that M is a reductive homogeneous manifold, and the canonical connection $\tilde{\nabla}$ associated with the reductive decomposition above (see [KN63, Vol. 2, Ch. X]) satisfies

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0.$$

Since the Lie group $A \times G$ leaves ξ invariant then $\tilde{\nabla}\xi = 0$ (see [CC19, p. 39, Prop. 1.4.15]).

To show that $\tilde{\nabla}_X g = 0$ for all $X \in \mathcal{D}$, we consider, for $X \in \mathfrak{m}$, the curve $\gamma(t) = \exp(tX) \cdot p$. Since Prop. 1.2.12, the parallel transport along the curve γ is the linear map given by the differential $(L_{\exp(tX)})_*\colon T_pM \longrightarrow T_{\gamma(t)}M$. As this map preserves the metric, we have that $\tilde{\nabla}_{\gamma} g = 0$. Finally, if we take X, Y two vector fields tangent to the leaves, since $\tilde{\nabla}_X g = 0$, we have $0 = g(\tilde{\nabla}_X Y, \xi) + g(Y, \tilde{\nabla}_X \xi) = g(\tilde{\nabla}_X Y, \xi)$. Hence the leaves are totally geodesic with respect to $\tilde{\nabla}$ and therefore $g(\tilde{T}(X,Y),\xi) = 0$.

The proof of (2) implies (1).

The completeness of $\tilde{\nabla}$ together with conditions $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$ and $\tilde{\nabla}\xi = 0$ implies (see Thm. 4.1.2) that M is a homogeneous manifold $M = \bar{G}/\bar{H}$, with reductive decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} + \bar{\mathfrak{m}}$, $\tilde{\nabla}$ is the canonical connection and the vector field ξ is $\bar{G}$-invariant. The Lie group $\bar{G}$ that acts transitively and effectively on $M = \bar{G}/\bar{H}$ is the Lie group of transvections of $\tilde{\nabla}$ (see Thm. 1.2.13). Since $\tilde{\nabla}_X g = 0$, for $X \in \mathcal{D}$, g is invariant along parallel transport by curves orthogonal to ξ.

Let $\bar{\mathfrak{g}}$ be the Lie algebra of $\bar{G}$ with reductive decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} + \bar{\mathfrak{m}}$. We can identify $\bar{\mathfrak{m}}$ with $T_{e\bar{H}}M$ via, $\psi(X) = \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot e\bar{H}$. As a consequence of this identification, we can endow $\bar{\mathfrak{m}}$ with a Riemannian metric structure g given by $g(X,Y) = g|_{e\bar{H}}(\psi X, \psi Y)$. Then, we can consider the subspace

$$\mathfrak{m} = \{X \in \bar{\mathfrak{m}} : g(X, \psi^{-1}\xi) = 0\} \subset \bar{\mathfrak{m}}$$

and the subalgebra

$$\mathfrak{h} = \operatorname{span}\{[X,Y]_{\bar{\mathfrak{h}}} : X, Y \in \mathfrak{m}\} \subset \bar{\mathfrak{h}}.$$

The algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is a Lie subalgebra of $\bar{\mathfrak{g}}$, and $\mathfrak{h}$ is a Lie subalgebra of $\bar{\mathfrak{h}}$. By Prop. 1.2.12, the geodesics of $\tilde{\nabla}$ starting from $e\bar{H}$ have the expression $\exp(tX) \cdot e\bar{H}$ for all $X \in \mathfrak{m}$ and all $t \in \mathbb{R}$. Indeed, parallel transport along such a geodesic is given by the differential map $(L_{\exp(tX)})_*$. Since $\tilde{\nabla}_X g = 0$, the transvection map $L_{\exp(tX)}$ is an isometry of M. From $[X,Y]_{\bar{\mathfrak{h}}} = \tilde{R}_{XY} = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}$ and $\tilde{\nabla}_X g = 0$, we conclude $\tilde{R}_{XY} \cdot g = 0$. In particular, the transvection map $L_{\exp(tH)}$ associated with each $H \in \mathfrak{h}$ is an isometry. Moreover, as $\tilde{R}_{XY} \cdot \xi = 0$ and $\tilde{R}_{XY} \cdot g = 0$, then $\mathfrak{m}$ is an $\mathfrak{h}$-module of $\mathfrak{g}$ and every leaf is a reductive Riemannian homogeneous manifold.

Let G be the connected Lie subgroup of $\bar{G}$ associated with $\mathfrak{g} \subset \bar{\mathfrak{g}}$. According to the above paragraph, G is a group of isometries of M.

Lemma 6.2.4. *M is a cohomogeneity one foliation of principal orbits associated with the action of G.*

Proof. The Lie group G preserves every leaf since its action preserves g and ξ, and G is connected. Since M is a connected homogeneous space, any two points p, $q \in M$ can be joined by a broken geodesic of $\tilde{\nabla}$ such that each geodesic is an integral curve of ξ or a $\tilde{\nabla}$-geodesic in a leaf (we are using here connectedness of M). Then, there exists $L_g = L_{g_1} \circ \varphi_{t_1} \circ \ldots \circ L_{g_m} \circ \varphi_{t_m}$ with $m \in \mathbb{N}$, $L_{g_i} \in G$ and φ the flow of ξ such that $L_g p = q$. As ξ is invariant for every L_{g_i}, then $L_g = \varphi_{t_1+\ \dots\ +t_m} \circ L_{g_1} \circ\ \dots\ \circ L_{g_m}$. The integral curves of ξ intersect all orbits. Therefore, to prove that G acts transitive on every leaf, it is enough to prove this fact in only one leaf.

Let N be the leaf of $e\bar{H}$. Since N is a totally geodesic submanifold of M, by Lem. 6.2.1, every point $p \in N$ is connected to $e\bar{H}$ by a broken geodesic of $\tilde{\nabla}$. As G is connected and locally transitive on an open neighbourhood of $e\bar{H}$ (in the submanifold topology), G acts transitively on N. Finally, every orbit is principal since $\{g \in G : g \cdot e\bar{H} = e\bar{H}\} = \{g \in G : g \cdot \varphi_t(e\bar{H}) = \varphi_t(e\bar{H})\}$, for all $t \in \mathbb{R}$ and ξ intersects all orbits. $\square$

This finishes the proof of Theorem 6.2.2. $\blacksquare$

Definition 6.2.5. A Riemannian manifold (M,g) equipped with a unit vector field ξ and a linear connection $\tilde{\nabla}$ satisfying conditions (6.1) is called a *cohomogeneity one Ambrose-Singer manifold* (CO1-AS-manifold for short). In this case, the $(1,2)$-tensor field $S = \nabla - \tilde{\nabla}$ is called *cohomogeneity one structure*, where ∇ is the Levi-Civita connection.

Remark 6.2.6. As we have described in Sec. 1.3.1, a same reductive homogeneous manifold may have different descriptions as a quotient G/H as well as different reductive decompositions $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Furthermore, for each of the quotient we may determine the set of all G-invariant metrics. Together with the choice of the complement $\mathfrak{m}$ to $\mathfrak{h}$, all these provide the collection of homogeneous tensors S.

A similar situation happens in cohomogeneity one manifolds. For example, for any Lie group G acting on $(\mathbb{S}^n, g)$ the Euclidean sphere, see (1.9). Then, we might consider a warped product $M = \mathbb{R} \times_f \mathbb{S}^n$ for a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$. Obviously, the Lie group $\{0\} \times G$ acts with cohomogeneity one on $\mathbb{R} \times_f \mathbb{S}^n$ and the action of every different group provides the same (trivial) foliation. Thus, the cohomogeneity one structure helps to distinguish between those different actions.

One advantage of AS or CO1-AS structures is that they can be used to distinguish different homogeneous or cohomogeneity one descriptions. With respect to CO1-AS structures, their classification is done up to equivalence in the following sense.

Definition 6.2.7. Two regular cohomogeneity one manifolds (M,g) and (M',g') with the same group G are called *isomorphic* if and only if there exists a G-equivariant isometry $f\colon (M,g) \longrightarrow (M',g')$.

Definition 6.2.8. Two cohomogeneity one structures S in (M,g) and S' in (M',g') are called *isomorphic* if and only if there exists an isometry $f\colon (M,g) \longrightarrow (M',g')$ that maps S to S'.

Proposition 6.2.9. *Let (M,g) and (M',g') be two connected, simply-connected and complete G-regular cohomogeneity one manifolds with cohomogeneity one structures S and S', respectively. If S and S' are isomorphic, then (M,g) and (M',g') are isomorphic.*

Proof. Let $f\colon (M,g) \longrightarrow (M',g')$ be an isometry sending S to S'. Then f is an affine map from $\tilde{\nabla} = \nabla - S$ to $\tilde{\nabla}' = \nabla' - S'$ where ∇ and ∇' are the Levi-Civita connections of (M,g) and (M',g'), respectively. We now consider the Lie group G generated by Thm. 6.2.2 applied for the first of the manifolds. It also acts on M' through f. We know from Thm. 1.2.13 that this Lie group is generated by the global transvections of the connection $\tilde{\nabla}$. For every global transvection F of $\tilde{\nabla}$, then $f \circ F \circ f^{-1}$ is a transvection map of $\tilde{\nabla}'$. Therefore, f is a G-equivariant map. □

Proposition 6.2.10. *Let (M,g) and (M',g') be two connected, simply-connected and complete G-regular cohomogeneity one manifolds. If (M,g) and (M',g') are isomorphic by $f\colon (M,g) \longrightarrow (M',g')$ and S is a cohomogeneity one structure for (M,g), then f_*S is a cohomogeneity one structure for (M',g) that is isomorphic to S.*

Proof. This proof is direct after the observation that f is an affine diffeomorphism between the connections $\tilde{\nabla} = \nabla - S$ and $\tilde{\nabla}' = \nabla' - f_*S$, where ∇ and ∇' are the Levi-Civita connections of (M,g) and (M',g'), respectively. □

6.3 Characterization of cohomogeneity one manifolds: local version

Let $\mathcal{G}$ be a Lie pseudo-group of differentiable transformations on a manifold M (we refer the reader to [Spi92] or [Acc21] for an exposition on this topic). Given a point $p_0 \in M$, we define $\mathcal{G}(p_0)$ as the set of transformations for which p_0 belongs to the domain, and $\mathcal{G}(p_0,p_0) \subset \mathcal{G}(p_0)$ the set of transformations f such that $f(p_0) = p_0$. The quotient $H(p_0) = \mathcal{G}(p_0,p_0)/\sim$ with respect to the relation $f \sim f' \iff f|_U = f'|_U$ for some neighbourhood U of p_0, is a Lie group (cf. [Acc21, Ch. 1]). We now consider a frame $u_0 \in \mathcal{L}(M)$ over the point p_0. We recall that the action of $\mathcal{G}$ on M is effective and closed if the map (4.2) is a monomorphism and its image $\mathbf{H}(u_0)$ is closed. The morphism (4.2) is called the isotropy representation of $\mathcal{G}$ on M. The effectiveness and closedness of this representation do not depend on the choice of u_0 (see Prop. 4.2.2). Following the construction in Sec. 4.2, an effective and closed action of $\mathcal{G}$ on M naturally induces an action of $\mathcal{G}$ on the frame bundle $\mathcal{L}(M)$:

$$(f,u) \mapsto f_{*,\pi(u)} \circ u, \quad u \in \mathcal{L}(M),$$

where $\mathbf{H}(u_0) \subset \mathrm{GL}(n,\mathbb{R})$ represents the isotropy group at u_0 for the Lie groupoid defined by the germs of differentials of $\mathcal{G}$. It is called the *linear isotropy group*.

Since two local isometries f, f' such that $f(p_0) = f'(p_0)$ coincide on a neighbourhood of p_0 (i. e., $f \sim f'$) if and only if their differentials at p_0 are equal, we have that the action of a pseudo-group of local isometries of a Riemannian manifold (M,g) is always effective.

If the action is both effective and transitive, given $u_0 \in \mathcal{L}(M)$, the bundle

$$\mathcal{P}(u_0) = \{f_*(u_0) : f \in \mathcal{G}\},$$

is a reduction of the frame bundle to the subgroup $\mathbf{H}(u_0)$. Under these conditions, for an element $\varphi \in H(p_0)$, we define

$$\begin{aligned} \mathrm{Ad}_\varphi \colon T_{u_0}P(u_0) &\longrightarrow T_{u_0}P(u_0) \\ \frac{d}{dt}\Big|_{t=0}(\varphi_t)_*(u_0) &\longmapsto \frac{d}{dt}\Big|_{t=0}(\varphi\circ\varphi_t\circ\varphi^{-1})_*(u_0) \end{aligned}$$

where $\varphi_t \in \mathcal{G}$, and t belongs to a certain interval $(-\varepsilon,\varepsilon)$.

Consider a Lie pseudo-group $\mathcal{G}$ whose action is transitive, effective and closed. In this context, we defined the action to be *reductive* (see Def. 4.2.3) if and only if the tangent space at u_0, admits a decomposition $T_{u_0}P(u_0) = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra associated with $\mathbf{H}(u_0)$ and $\mathfrak{m}$ is a $\mathrm{Ad}(H(p_0))$-invariant subspace.

Theorem 6.3.1 ([CC19, p. 3.1.13])**.** *Let (M,g) be a Riemannian manifold such that a pseudo-group $\mathcal{G} \subset \mathrm{Isom}_{loc}(M,g)$ is acting transitively. Then, the action of $\mathcal{G}$ is effective, closed and reductive.*

Definition 6.3.2. A Riemannian manifold (M,g) is *locally cohomogeneity one* if there exists a pseudo-group of local isometries $\mathcal{G}$ acting on M in such a way that $M/\mathcal{G}$ is connected and every orbit is an orientable, embedded and closed submanifold of codimension one.

A locally cohomogeneity one Riemannian manifold (M,g) is said to be a *regular* if for every two frames u_0 and v_0 of M, the closed subgroups $\mathbf{H}(u_0)$ and $\mathbf{H}(v_0)$ are conjugate in $\mathrm{GL}(n,\mathbb{R})$. This is equivalent to the fact that all orbits are principal.

Theorem 6.3.3. *Let $(M,\, g)$ be an orientable and connected Riemannian manifold. The following two are equivalent:*

(1) *(M,g) is a regular locally cohomogeneity one Riemannian manifold.*

(2) *There exists a connection $\tilde\nabla$ and a unit vector field $\xi \in \mathfrak{X}(M)$ such that*

$$\begin{aligned} &\tilde\nabla\tilde R = 0, \quad \tilde\nabla\tilde T = 0, \quad \tilde\nabla\xi = 0, \\ &\tilde\nabla_X g = 0, \quad \tilde T(X,Y) \in \mathcal{D}, \qquad \forall X,Y \in \mathcal{D}, \end{aligned} \tag{6.3}$$

where $\mathcal{D} = \{X \in \mathfrak{X}(M) : \, g(X,\xi) = 0\}$, and $\tilde R$, $\tilde T$ are the curvature and torsion of $\tilde\nabla$, respectively. Furthermore, the maximal integral leaves of the distribution $\mathcal{D}$ (which is integrable according to Lem. 6.2.1) are embedded and closed.

Proof. The main ideas of the proof are summarized in Thm. 6.2.2.

Proof of (2) implies (1). From Thm. 4.2.5 we know that there exists reductive Lie pseudo-group $\bar{\mathcal{G}}$ that acts transitively on (M,g) and leaves ξ invariant. Moreover, this pseudo-group $\bar{\mathcal{G}}$ can be identified with the Lie pseudo-group of local transvections of $\tilde{\nabla}$. We thus consider the subset $\mathcal{G}$ of $\bar{\mathcal{G}}$ consisting of local transvections associated with parallel transports by curves in the leaves. Since $\tilde{\nabla}\xi = 0$ and $\tilde{\nabla}_X g = 0$ for all $X \in \mathcal{D}$, these local transvections preserve the vector field ξ, the Riemannian tensor g and thus the codimension one distribution $\mathcal{D}$.

Lemma 6.3.4. *The subset $\mathcal{G}$ is a Lie pseudo-group.*

Proof. This follows directly from the definition of local transvection, i. e., it is a local diffeomorphism that preserves every holonomy bundle. □

Lemma 6.3.5. *For any $f \in \mathcal{G}$ and any point $q \in \operatorname{dom}(f)$, both q and $f(q)$ belong to the same leaf. Furthermore, the action of $\mathcal{G}$ on every leaf is transitive.*

Proof. The local transvection $f \in \mathcal{G}$ is associated with a $\tilde{\nabla}$-parallel transport along a broken geodesic in one leaf O, and that leaf is thus invariant. But the point $q \in \operatorname{dom}(f)$ does not have to belong to that leaf. In that case, there exists $t_0 \in \mathbb{R}$ such that $\varphi_{t_0}(q) \in O$, where φ_t is the flow of ξ. The vector field is geodesic for $\tilde{\nabla}$, the differential of the flow is the $\tilde{\nabla}$-parallel transport along ξ and the flow sends leaves to leaves. As f preserves ξ, the points q and $f(q)$ belong to the same leaf $\varphi_{-t_0}(O)$.

The transitiveness comes from the existence of (broken) geodesics connecting any two point of any leaf. □

Proof of (1) implies (2). The procedure is to find a connection $\tilde{\nabla}$ satisfying the first row of (6.3). Again, from Thm. 4.2.5, it is sufficient to show that there exist a Lie pseudo-group $\bar{\mathcal{G}}$ whose action is transitive (Lem. 6.3.6), effective (Lem. 6.3.7), closed (Lem. 6.3.8) and reductive (Lem. 6.3.9) on the leaves. Afterwards, we prove the second row of (6.3).

Since M is orientable and all orbits are principal, any orbit admits a unit normal vector field ξ. As M/G is connected, in analogy with Thm. 6.2.2, the vector field ξ is globally defined on M. That is, for every orbit $\mathcal{G}\cdot p_o$ there exists an $\varepsilon > 0$ such that the flow $\varphi\colon (-\varepsilon,\varepsilon)\times U \longrightarrow M$, where U is an open set containing $\mathcal{G}\cdot p_0$, is defined. Let $\mathcal{G}$ be the Riemannian pseudo-group acting with cohomogeneity one on M. We define the pseudo-group $\bar{\mathcal{G}}$ generated by $\mathcal{G}$ and $\mathcal{A}$, the latter being the pseudo-group generated by flows of unit local vector fields orthogonal to the leaves.

Lemma 6.3.6. *The Lie pseudo-group $\bar{\mathcal{G}}$ acts transitively on M.*

Proof. Let p and q be two points. If p and q belong to the same leaf, then there exists a local transformation $f \in \mathcal{G}$ such that $f(p) = q$. In general, it is easy to see that $A = \{x \in N : f(p) = x, f \in \bar{\mathcal{G}}\}$ is open and closed. Since M is connected, we get the result. $\square$

As ξ is invariant by the isometries of $\mathcal{G}$, every geodesic flow orthogonal to the leaves commutes with the local isometries of $\mathcal{G}$ by Lem. 6.2.3. Actually, every transformation $\bar{f} \in \bar{\mathcal{G}}$ can be written as $\bar{f} = \varphi \circ f$, where $\varphi \in \mathcal{A}$ and $f \in \mathcal{G}$.

Lemma 6.3.7. *The Lie pseudo-group $\bar{\mathcal{G}}$ is effective.*

Proof. According to the first paragraph of Section 6.3, let $H_{\mathcal{G}}(p_0)$ be the quotient space of elements $f \in \mathcal{G}$ fixing a point p_0. Analogously, the space of elements $\bar{f} \in \bar{\mathcal{G}}$ fixing a point p_0 is given by

$$H(p_0) = H_{\mathcal{G}}(p_0) + \mathcal{L}\left(\varphi_{t_0} \circ f^{-1}\right),$$

where $\mathcal{L}(\varphi_{t_0} \circ f^{-1})$ is the discrete Lie group generated by the element $\varphi_{t_0} \circ f^{-1}$ and t_0 is the smallest $t_0 > 0$ such that $\varphi_{t_0}(p_0) = f(p)$ for some $f \in \mathcal{G}$. Note that, the set $D = \{\varphi_{kt_0}(p_0) : k \in \mathbb{Z}\}$ is the intersection of $\{\varphi_t(p_0) : t \in \mathbb{R}\}$ and $\mathcal{G} \cdot p_0$, and both subsets are closed in N.

Note that $H_{\mathcal{G}}(p_0)$ is effective because it arises from a Riemannian Lie pseudo-group. We show that if $(\varphi_{t_0} \circ f^{-1})_*(p_0) = Id_{T_{p_0}M}$, then $\varphi_{t_0} \circ f^{-1}$ is the identity map in an open neighbourhood U of M. There is an open neighbourhood U such that both domain and image of $\varphi_{t_0} \circ f^{-1}$ are contained in U. This map is $\mathcal{G}(U)$-equivariant by local isometries $h \in \mathcal{G}$ with domain U, i. e., for every $h \in \mathcal{G}(U)$ there exits $h' \in \mathcal{G}(U)$ such that $\varphi_{t_0} \circ f^{-1} \circ h = h' \circ \varphi_{t_0} \circ f^{-1}$. For every local Killing vector field X^*, its flow comes from a family F_t of isometries with $t \in I$, where I is a closed interval, and $F_t(p_0)$ is an integral curve of X^*. Due to the equivariance, there exists a local Killing vector field Y^* and a family H_t of isometries such that,

$$\varphi_{t_0} \circ f^{-1} \circ F_t = H_t \circ \varphi_{t_0} \circ f^{-1}.$$

When applied to p_0, we get

$$\varphi_{t_0} \circ f^{-1} \circ F_t(p_0) = H_t(p_0),$$

i. e., $(\varphi_{t_0} \circ f^{-1})_*(X^*(p_0)) = Y^*(p_0)$. Necessarily, $X^* = Y^*$; therefore $H_t = F_t$. Consequently, by applying this argument for every local Killing vector field, we conclude that $(\varphi_{t_0} \circ f^{-1})$ fixes integral curves of Killing vector fields with the initial point p_0. In these conditions $\varphi_{t_0} \circ f^{-1} = Id_U$. $\square$

Lemma 6.3.8. *The action of the Lie pseudo-group $\bar{\mathcal{G}}$ is closed.*

Proof. The image of $H(p_0)$ by the isotropy map (4.2) is

$$\mathbf{H}(u_0) = \mathbf{H}_{\mathcal{G}}(u_0) + \mathcal{L}({u_0}^{-1} \circ (\varphi_{t_0} \circ f^{-1})_* \circ u_0)$$

where $\mathbf{H}_{\mathcal{G}}(u_0)$ is the image of $H_{\mathcal{G}}(p_0)$. On the one hand, as every leaf $\mathcal{G} \cdot p$ is a locally homogeneous Riemannian manifold and its action is reductive (see [Tri92]), then $\mathbf{H}_{\mathcal{G}}(u_0)$ is equal to the holonomy group of one AS-connection. Consequently, it is closed in $\mathrm{GL}(n, \mathbb{R})$.

On the other hand, let $A = {u_0}^{-1} \circ (\varphi_{t_0} \circ f^{-1})_* \circ u_0 \in \mathrm{GL}(n, \mathbb{R})$. Then, the space $\mathcal{L}(A)$ consists of the powers A^k for $k \in \mathbb{Z}$. The matrix $A \in \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{C})$ diagonalizes as $A = P \cdot D \cdot P^{-1}$. We consider a convergent sequence $\{A^{k_n}\}$ in $\mathrm{GL}(n, \mathbb{C})$. Its eigenvalues cannot have a norm different from 1; otherwise the limit would have a singular eigenvalue. If the eigenvalues are of the form e^{iv} with v irrational, $\{A^{k_n}\}$ would not converge. Then the exponents of the eigenvalues are rational and the group $\mathcal{L}(A)$ is cyclic, finite and closed. Finally, since $\mathbf{H}_{\mathcal{G}}(u_0)$ and $\mathcal{L}(A)$ are closed, it follows that $\mathbf{H}(u_0)$ is closed. □

As $\mathcal{L}((\varphi_{t_0} \circ g^{-1}))$ is discrete, the Lie algebra of $H(p_0)$ is equal to the Lie algebra $\mathfrak{h}$ of $\mathrm{H}_{\mathcal{G}}(p_0)$.

Lemma 6.3.9. *The Lie pseudo-group $\bar{\mathcal{G}}$ acts reductively.*

Proof. Given a point $p_0 \in M$ and a frame u_0 on it, we consider

$$\mathcal{P}(u_0) = \left\{ f_* \circ u_0 \in \mathcal{L}(M) : f \in \bar{\mathcal{G}} \right\}$$

and $\mathcal{P}(u_0, \mathcal{G} \cdot p_0) = \{f_* \circ u_0 \in \mathcal{L}(\mathcal{G} \cdot p_0) : f \in \mathcal{G}\}$, the reduction of the frame bundle of M and the orbit $\mathcal{G} \cdot p_0$, respectively. Since every locally homogeneous Riemannian manifold is reductive (see Thm. 6.3.1 above), we have the reductive decomposition in the leaves: $T_{u_0}\mathcal{P}(u_0, \mathcal{G} \cdot p_0) = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m}$ is an $\mathrm{Ad}(H_{\mathcal{G}}(p_0))$-invariant subspace.

We consider the decomposition $T_{u_0}P(u_0) = \mathfrak{h} + (\mathfrak{a} + \mathfrak{m})$, where

$$\mathfrak{a} = \left\{ \frac{d}{dt}\Big|_{t=0} (\varphi)_* \circ u_0 : \varphi \in \mathcal{A} \right\}.$$

As every unit vector field ξ orthogonal to the leaves is invariant under the local transformations of $\mathcal{G}$, the flow of ξ commutes with every $f \in \mathcal{G}$. Consequently, $\mathfrak{a}$ is $\mathrm{Ad}(\mathrm{H}_{\mathcal{G}}(p_0))$-invariant and $\mathrm{Ad}(\varphi_{t_0} \circ L_{g^{-1}})$-invariant. Finally, as $\varphi_{t_0} \circ L_{g^{-1}}$ preserves ξ and the horizontal distribution defined by the connection above, the subspace $\mathfrak{m}$ is $\mathrm{Ad}(\varphi_{t_0} \circ L_{g^{-1}})$-invariant. □

Therefore, from Thm. 4.2.5, there exists a global connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}\tilde{R}=0, \quad \tilde{\nabla}\tilde{T}=0.$$

Finally, the proof of

$$\tilde{\nabla}\xi=0, \quad \tilde{\nabla}_X g=0, \quad \tilde{T}(X,Y)\in\mathcal{D}$$

is analogous to the corresponding proof in Thm. 6.2.2. ■

We also have the following properties of CO1-AS-manifolds.

Proposition 6.3.10. *Let $(M,g,\tilde{\nabla})$ be a CO1-AS-manifold and let $S=\nabla-\tilde{\nabla}$ be the corresponding cohomogeneity one structure tensor.*

1. *For every $X\in\mathcal{D}$, we have $\tilde{\nabla}_X S=0$.*
2. *For every $X\in\mathcal{D}$, $S_X\cdot g=0$.*
3. *For every X, $Y\in\mathcal{D}$ then $g(S_XY,\xi)=g(\nabla_XY,\xi)=g(\mathrm{II}(X,Y),\xi)$, where II is the second fundamental form of the leaves.*
4. *We have that $S_\xi\xi=0$.*

Proof. **1.** Since $\tilde{\nabla}_X g=0$, every parallel transport defined by $\tilde{\nabla}$ along curves belonging to one leaf preserves the Levi-Civita connection and the connection ∇. Therefore, they also preserve the difference tensor $S=\nabla-\tilde{\nabla}$, that is, $\tilde{\nabla}_X S=0$.

2. From $\tilde{\nabla}_X g=0$ and $\nabla g=0$, we have $(\nabla-\tilde{\nabla})_X g=S_X g=0$.

3. First, $g(S_XY,\xi)=g((\nabla-\tilde{\nabla})_XY,\xi)=g(\nabla_XY,\xi)-g(\tilde{\nabla}_XY,\xi)$. From $\tilde{\nabla}_X g=0$, this is equal to $g(\nabla_XY,\xi)+g(Y,\tilde{\nabla}_X\xi)=g((\nabla_XY)^{\perp},\xi)=g(\mathrm{II}(X,Y),\xi)$.

4. It is a consequence of the fact that ξ is a geodesic vector field and $\tilde{\nabla}\xi=0$. □

6.4 Decomposition of cohomogeneity one Structures

We now explore the infinitesimal models associated with CO1-AS manifolds as well as their classification.

Let (V,g) be a vector space of dimension $n+1$, $n\in\mathbb{N}$, endowed with a positive definite inner product g and a unit vector ξ. Let

$$\tilde{R}\colon V\wedge V\longrightarrow \mathrm{End}(V), \quad \tilde{T}\colon V\longrightarrow \mathrm{End}(V),$$

be two linear homomorphisms. We say that $(\tilde{R},\tilde{T})$ is an *infinitesimal cohomogeneity one model* if it satisfies

$$\begin{aligned}
&\tilde{T}_X Y + \tilde{T}_Y X = 0,\\
&\tilde{R}_{XY} Z + \tilde{R}_{YX} Z = 0,\\
&\tilde{R}_{XY}\cdot\tilde{T} = \tilde{R}_{XY}\cdot\tilde{R} = 0,\\
&\underset{XYZ}{\mathfrak{S}}\tilde{R}_{XY}Z + \tilde{T}_{\tilde{T}_X Y}Z = 0,\\
&\underset{XYZ}{\mathfrak{S}}\tilde{R}_{\tilde{T}_X YZ} = 0,\\
&\tilde{R}_{XY}\cdot\xi = 0,
\end{aligned}$$

and, for all $X, Y \in D = \mathcal{L}(\xi)^{\perp}$,

$$\begin{aligned}
&\tilde{R}_{XY}\cdot g = 0,\\
&\tilde{R}_{XY}\cdot S = 0,\\
&\tilde{T}_X Y \in D,
\end{aligned}$$

where $\underset{XYZ}{\mathfrak{S}}$ is the cyclic sum, and the dot stands for the action of $\tilde{R}_{XY}$ on V as a derivation.

Two CO1-AS infinitesimal models $(V,\tilde{T},\tilde{R},\xi,g,S)$, $(V,\tilde{T}',\tilde{R}',\xi',g',S')$ are said to be isomorphic if there is an isometry

$$f\colon (V,g) \longrightarrow (V',g')$$

such that,

$$f\tilde{R} = \tilde{R}', \quad f\tilde{T} = \tilde{T}', \quad f(\xi) = \xi, \quad fS = S'.$$

Remark 6.4.1. In particular, at any point p of a CO1-AS manifold M there is an infinitesimal cohomogeneity one model by taking $V = T_pM$, g_p, ξ_p, $\tilde{R}_p$ and $\tilde{T}_p$. According to Thm. 6.3.3, two points p, p' in the same leaf of the foliation of M define isomorphic infinitesimal cohomogeneity one models.

Given an infinitesimal cohomogeneity one model, we define

$$\mathcal{S}(V) = \{S \in V^* \otimes \mathfrak{gl}(V) : S_X\cdot g = 0, S_\xi \xi = 0, \forall X \in D\}$$

as the *space of infinitesimal cohomogeneity one structures*. From the decomposition $V = U \oplus D$ as irreducible SO(n)-submodules, $U = \mathcal{L}(\xi)$, $D = U^{\perp}$, we have

$$\mathcal{S}(V) = \mathcal{S}_D(V) + \mathcal{S}_U(V)$$

with

$$\mathcal{S}_D(V) = \{S \in D^* \otimes \mathfrak{gl}(V) : S_X \cdot g = 0\} \simeq D^* \otimes \mathfrak{so}(V) = D^* \otimes V \wedge V,$$
$$\mathcal{S}_U(V) = \{S \in U^* \otimes \mathfrak{gl}(V) : S_\xi \xi = 0\}.$$

We now explore each of these $\mathrm{SO}(n)$-submodules. As usual, we can identify spaces and their duals through the metric g.

We begin with $\mathcal{S}_D(V)$. It can be decomposed in $\mathrm{SO}(n)$-submodules as

$$\begin{aligned}\mathcal{S}_D(V) &= D^* \otimes V^* \wedge V^* \\ &= D^* \otimes D^* \wedge D^* + D^* \otimes U^* \otimes D^*,\end{aligned}$$

decomposition that is denoted by

$$\mathcal{T}(V) = D^* \otimes D^* \wedge D^*,$$
$$\mathcal{II}(V) = D^* \otimes U^* \otimes D^*.$$

As we will see when we understand these structures coming from a CO1-AS manifold, this notation is motivated by the fact that $\mathcal{T}(V)$ is the space of all possible infinitesimal homogeneous structures tensors of the leaves and $\mathcal{II}(V)$ is the space of all possible second fundamental forms of the leaves at each point. The decomposition of these modules into irreducible submodules can be derived from the results in the literature. They are as follows.

Proposition 6.4.2. *The submodule $\mathcal{T}(V)$ decomposes as $\mathrm{SO}(n)$-irreducible submodules as,*

$$\mathcal{T}(V) = \mathcal{T}_1(V) + \mathcal{T}_2(V) + \mathcal{T}_3(V)$$

with explicit expressions,

$$\mathcal{T}_1(V) = \Big\{S \in \mathcal{T}(V) : S_{XYZ} = g(X,Y)\theta(Z) - g(X,Z)\theta(Y),\, \theta \in D^*\Big\},$$
$$\mathcal{T}_2(V) = \Big\{S \in \mathcal{T}(V) : \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0,\, c_{12}(S) = 0\Big\},$$
$$\mathcal{T}_3(V) = \Big\{S \in \mathcal{T}(V) : S_{XYZ} + S_{YXZ} = 0\Big\},$$

where $c_{12}(S)(Z) = \sum_{i=1}^n S_{e_ie_iZ}$ for any orthonormal basis $\{e_1, \ldots, e_n\}$ of D.

Proof. The proof follows directly from [TV83, Thm. 3.1] applied to our notation. □

Proposition 6.4.3. *The submodule $\mathcal{II}(V)$ decomposes as* $\mathrm{SO}(n)$*-irreducible submodules as,*

$$\mathcal{II}(V) = \mathcal{II}_1(V) + \mathcal{II}_2(V) + \mathcal{II}_3(V)$$

with explicit expressions,

$$\begin{aligned}
\mathcal{II}_1(V) &= \Big\{ S \in \mathcal{II}(V) : S_{XYZ} = \lambda g(X,Z) g(Y,\xi),\, \lambda \in \mathbb{R} \Big\},\\
\mathcal{II}_2(V) &= \Big\{ S \in \mathcal{II}(V) : S_{XYZ} = g(Y,\xi) S_{X\xi Z},\, S_{X\xi Z} = S_{Z\xi X},\, c_{13}(S) = 0 \Big\},\\
\mathcal{II}_3(V) &= \Big\{ S \in \mathcal{II}(V) : S_{XYZ} = g(Y,\xi) S_{X\xi Z},\, S_{X\xi Z} = -S_{Z\xi X} \Big\},
\end{aligned}$$

where $c_{13}(S)(Y) = \sum_{i=1}^{n} S_{e_i Y e_i}$ for any orthonormal basis $\{e_1, \ldots, e_n\}$ of D.

Proof. Recall that $\mathcal{II}(V) = D^* \otimes U^* \otimes D^*$. Indeed, for any $S \in D^* \otimes U^* \otimes D^*$, we have $S_{XYZ} = g(Y,\xi) S_{X\xi Z}$, and $S_{\cdot\,\xi\,\cdot} \in D^* \otimes D^*$. This space decomposes into irreducible $\mathrm{SL}(D)$ submodules as

$$\mathcal{II}(V) = D^* \wedge D^* + S^2 D^*,$$

in symmetric and skew-symmetric endomorphisms, respectively. To get the $\mathrm{SO}(D) \subset \mathrm{SL}(D)$ irreducible submodules we have to take traces in the endomorphisms, i. e.,

$$\mathcal{II}(V) = D^* \wedge D^* + \{ S_{X\xi Z} \in S^2 D^* : \mathrm{c}_{12}(S)(\xi) = 0 \} + \{ S_{X\xi Z} \in S^2 D^* : \mathrm{c}_{12}(S)(\xi) \neq 0 \}$$

and this decomposition into irreducible submodules yields the three expressions $\mathcal{II}_3(V)$, $\mathcal{II}_2(V)$ and $\mathcal{II}_1(V)$, respectively. □

We now analyse

$$\mathcal{S}_U(V) = \{ S \in U^* \otimes \mathfrak{gl}(V) : S_\xi \xi = 0 \} = U^* \otimes V^* \otimes V.$$

From $V = U \oplus D$ we get

$$U^* \otimes V^* \otimes V^* = U^* \otimes U^* \otimes V^* + U^* \otimes D^* \otimes D^* + U^* \otimes D^* \otimes U^*$$

and since $S_\xi \xi = 0$, the space $\mathcal{S}_U(V)$ is exactly

$$\mathcal{S}_U(V) = U^* \otimes D^* \otimes D^* + U^* \otimes D^* \otimes U^*,$$

decomposition that is denoted by

$$\mathcal{Z}(V) = U^* \otimes D^* \otimes D^*,$$
$$S^1_U(V) = U^* \otimes D^* \otimes U^*.$$

Proposition 6.4.4. *The submodule $\mathcal{Z}(V)$ decomposes as* $\mathrm{SO}(n)$*-irreducible submodules as,*

$$\mathcal{Z}(V) = \mathcal{Z}_1(V) + \mathcal{Z}_2(V) + \mathcal{Z}_3(V)$$

with explicit expressions,

$$\mathcal{Z}_1(V) = \Big\{ S \in \mathcal{Z}(V) : S_{XYZ} = \lambda g(Y,Z) g(X,\xi),\, \lambda \in \mathbb{R} \Big\},$$
$$\mathcal{Z}_2(V) = \Big\{ S \in \mathcal{Z}(V) : S_{XYZ} = g(X,\xi) S_{\xi YZ},\, S_{\xi YZ} = S_{\xi YX},\, c_{23}(S) = 0 \Big\},$$
$$\mathcal{Z}_3(V) = \Big\{ S \in \mathcal{Z}(V) : S_{XYZ} = g(X,\xi) S_{\xi YZ},\, S_{\xi YZ} = -S_{\xi ZY} \Big\},$$

where $c_{23}(S)(X) = \sum_{i=1}^n S_{Xe_ie_i}$ for any orthonormal basis $\{e_1, \dots, e_n\}$ of D.

Proof. As $\mathcal{Z}(V) \simeq \mathcal{II}(V)$, this decomposition is analogous to Thm. 6.4.3 doing a permutation in the indices. □

Proposition 6.4.5. *The submodule $S^1_U(V)$ is* $\mathrm{SO}(D)$*-irreducible and its explicit expression is,*

$$\mathcal{S}^1_U(V) = \Big\{ S \in \mathcal{S}(V) : S_{XYZ} = g(Y,\eta) g(X,\xi) g(Z,\xi),\, \eta \in D \Big\}.$$

Proof. It is direct from the fact $S^1_U(V) = U^* \otimes D^* \otimes U^* \simeq D^*$ and it is irreducible. Its explicit expression is direct. □

Therefore, we can write the space of cohomogeneity one structures as

$$\mathcal{S}(V) = \mathcal{T}(V) + \mathcal{II}(V) + \mathcal{Z}(V) + S^1_U(V).$$

From the previous four propositions we have that, in fact, $\mathcal{S}(V)$ decomposes into the sum of ten irreducible $\mathrm{SO}(n)$-submodules. The local isometries in CO1-AS manifolds in Thm. 6.3.3 give the following result.

Proposition 6.4.6. *If the cohomogeneity one structure S_p of a CO1-AS manifold (M,g) at a point $p \in M$ belongs to a certain irreducible submodule or sum of irreducible submodules, then $S_{p'}$ belongs to the same submodule or sum of submodules for any other point p' of the leaf of p.*

We now give a few simple geometric results stemming from the classification above. It is easy to see that the projections to each submodule are

$$\begin{aligned}
\Pi_{\mathcal{T}(V)}(S)_{XYZ} &= \sum_{i,j=0}^{n} g(X,e_i)g(Y,e_j)S_{e_ie_jZ},\\
\Pi_{\mathcal{II}(V)}(S)_{XYZ} &= \sum_{i=0}^{n} g(X,e_i)g(Y,\xi)S_{e_i\xi Z},\\
\Pi_{\mathcal{Z}(V)}(S)_{XYZ} &= \sum_{i=0}^{n} g(Y,e_i)g(X,\xi)S_{e_i\xi Z},\\
\Pi_{S^1_U(V)}(S)_{XYZ} &= g(X,\xi)g(Z,\xi)S_{\xi Y\xi}.
\end{aligned}$$

Proposition 6.4.7. *Let (M,g,ξ) be a CO1-AS manifold and $p\in M$.*

If $\Pi_{\mathcal{T}(V)}(S_p)_{XYZ}=0$, then the leaf of p is a locally symmetric Riemannian manifold.

If $\Pi_{\mathcal{T}(V)}(S_p)_{XYZ}\in\mathcal{T}_1(V)$, then the leaf of p is locally isometric to the real hyperbolic space.

If $\Pi_{\mathcal{II}(V)}(S_p)_{XYZ}=0$, then the leaf of p is a totally geodesic submanifold.

6.5 The canonical cohomogeneity one structure

On the one hand, in the proof (1) implies (2) of the classical Ambrose-Singer Theorem, the procedure is to construct the canonical connection (see Def. 1.2.11) associated with the reductive decomposition given. As we have described in Sec. 1.2.1 this connection is characterized from an algebraic (see Thm. 1.2.10) and geometrical (see Prop. 1.2.12) point of view. On the other hand, in this section we give a formula for the cohomogeneity one structure constructed in the proof (1) implies (2) of Thm. 6.2.2 along a geodesic that intersects all the orbits of a cohomogeneity one action without singular orbits.

We consider a reductive homogeneous space $M=G/H$ (not necessarily Riemannian), with reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. For any $X\in\mathfrak{g}$ we denote by X^* the fundamental vector field associated with X. The canonical connection $\tilde\nabla$ of M with respect to the above decomposition is determined by (1.6)

$$\left(\tilde\nabla_{X^*}Y^*\right)_o=-[X,Y]_{\mathfrak{m}},$$

for any X, $Y\in\mathfrak{m}$, $o=eH$. Since Thm. 1.2.15, if $\tilde T$ is the torsion of $\tilde\nabla$, then we also have $\tilde T_o(X^*_o,Y^*_o)=-[X,Y]_{\mathfrak{m}}$ for all X, $Y\in\mathfrak{m}$.

Lemma 6.5.1. *If $X\in\mathfrak{m}$ and $Y\in\mathfrak{g}$ then $\tilde\nabla_X Y^*=-[X,Y]_{\mathfrak{m}}$.*

Proof. Using the formulas for $\tilde{\nabla}$ and the torsion $\tilde{T}_o$, when the corresponding vector are in $\mathfrak{m}$, we have

$$\begin{aligned}
\tilde{\nabla}_X Y^* &= \tilde{\nabla}_{X_o^*} Y^* = \left(\tilde{\nabla}_{X^*} Y^*\right)_o \\
&= \left(\tilde{\nabla}_{Y^*} X^*\right)_o + [X^*, Y^*]_o + \tilde{T}_o(X^*, Y^*) \\
&= \tilde{\nabla}_{Y_o^*} X^* - [X, Y]_o^* + \tilde{T}_o(X_o^*, Y_o^*) \\
&= \tilde{\nabla}_{Y_\mathfrak{m}} X^* - [X, Y]_\mathfrak{m} + \tilde{T}_o(X, Y_\mathfrak{m}) \\
&= -[Y_\mathfrak{m}, X]_\mathfrak{m} - [X, Y]_\mathfrak{m} - [X, Y_\mathfrak{m}]_\mathfrak{m} = -[X, Y]_\mathfrak{m},
\end{aligned}$$

as we wanted to show. □

Now we calculate $\tilde{\nabla}$ at any other point.

Lemma 6.5.2. *If X, $Y \in \mathfrak{m}$ and $f \in G$, we have*

$$\left(\tilde{\nabla}_{X^*} Y^*\right)_{f(o)} = -f_{*o}\left[\left(\mathrm{Ad}(f^{-1})X\right)_\mathfrak{m}, \mathrm{Ad}(f^{-1})Y\right]_\mathfrak{m}.$$

Proof. Let $X \in \mathfrak{g}$. We consider $\psi_t = \exp(tX)$ the 1-parameter group generated by X. We denote by $I_{f^{-1}}$ the conjugation by f^{-1}, and by $\mathrm{Ad}(f^{-1}) = I_{f^{-1}*}$ the corresponding adjoint representation of $\mathfrak{g}$. We have

$$X^*_{f(o)} = \frac{d}{dt}\Big|_{t=0} \psi_t(f(o)) = \frac{d}{dt}\Big|_{t=0} f I_{f^{-1}}(\psi_t)(o) = f_{*o}\left(\mathrm{Ad}(f^{-1})X\right)_o.$$

Hence, $\left(f_*^{-1} X^*\right)_o = \left(\mathrm{Ad}(f^{-1})X\right)_o$. Since $\tilde{\nabla}$ is G-invariant, Lem. 6.5.1 yields

$$\begin{aligned}
\left(\tilde{\nabla}_{X^*} Y^*\right)_{f(o)} &= f_{*o}\left(\tilde{\nabla}_{f_*^{-1} X^*} f_*^{-1} Y^*\right)_o = f_{*o} \tilde{\nabla}_{\left(\mathrm{Ad}(f^{-1})X\right)_\mathfrak{m}} \left(\mathrm{Ad}(f^{-1})Y\right)^* \\
&= f_{*o}\left[\left(\mathrm{Ad}(f^{-1})X\right)_\mathfrak{m}, \mathrm{Ad}(f^{-1})Y\right]_\mathfrak{m},
\end{aligned}$$

as claimed. □

Now we come back to our original problem. We consider a Riemannian manifold M that is being acted upon by a group of isometries G of cohomogeneity one, all whose orbits are principal. As we have seen, we can define a global unit vector field ξ that is orthogonal to all the orbits of G. We denote by φ_t the flow of ξ. Then φ_t determines a 1-parameter group A that is isomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. As a consequence, M is now a homogeneous space acted upon by $A \times G$ by the formula $(t, g) \cdot p = \varphi_t(g \cdot p) = g \cdot \varphi_t(p)$. In particular, A is contained in the center of $A \times G$, and ξ is the fundamental vector field on M determined by $1 \in A$.

From now on we fix $p \in M$. We have the reductive decomposition $\mathfrak{a}+\mathfrak{m} = \mathfrak{h} + (\mathfrak{a}+\mathfrak{m})$ at p. We determine the canonical connection $\tilde{\nabla}$ associated with this decomposition. We define $\gamma(t) = \varphi_t(p)$, which is a unit speed geodesic that intersects all the orbits of G orthogonally.

Proposition 6.5.3. *If X, $Y \in \mathfrak{m}$, then we have* $\big(\tilde{\nabla}_\xi X^*\big)_{\gamma(t)} = \big(\tilde{\nabla}_{X^*}\xi\big)_{\gamma(t)} = \big(\tilde{\nabla}_\xi \xi\big)_{\gamma(t)} = 0$, *and* $\big(\tilde{\nabla}_{X^*}Y^*\big)_{\gamma(t)} = -\varphi_{t*p}[X,Y]_{\mathfrak{m}}$.

Proof. Since A is contained in the center of $A\times G$ we have that $\mathrm{Ad}(t)$, $t\in A$, acts as the identity of $\mathfrak{a}+\mathfrak{g}$. Thus, $\mathrm{Ad}(t^{-1})\xi = \xi$ and $\mathrm{Ad}(t^{-1})X = X$ for all $X \in \mathfrak{m}$. Since $[\mathfrak{a},\mathfrak{a}+\mathfrak{g}] = 0$, the first three equalities of the statement follow from Lem. 6.5.2. Finally, if X, $Y \in \mathfrak{m}$, Lem. 6.5.2 yields

$$\begin{aligned}\big(\tilde{\nabla}_{X^*}Y^*\big)_{\gamma(t)} &= -\varphi_{t*p}\big[\big(\mathrm{Ad}(t^{-1}X)\big)_{\mathfrak{a}+\mathfrak{m}}, \mathrm{Ad}(t^{-1}Y)\big]_{\mathfrak{a}+\mathfrak{m}} \\ &= -\varphi_{t*p}\big[X_{\mathfrak{a}+\mathfrak{m}}, Y\big]_{\mathfrak{a}+\mathfrak{m}} = -\varphi_{t*p}\big[X,Y\big]_{\mathfrak{m}},\end{aligned}$$

which finishes the proof. □

Finally, we calculate the cohomogeneity one structure $S = \nabla - \tilde{\nabla}$, where ∇ is the Levi-Civita connection of M.

Proposition 6.5.4. *Let X, Y, $Z \in \mathfrak{m}$, and a, b, $c \in \mathbb{R}$. Then, the difference tensor S is given by*

$$\begin{aligned}&2g\big(S_{a\xi+X^*}(b\xi+Y^*), c\xi+Z^*\big)_{\gamma(t)} = \\ &\quad a\frac{d}{dt}g\big(\varphi_{t*p}Y,\varphi_{t*p}Z\big) + b\frac{d}{dt}g\big(\varphi_{t*p}X,\varphi_{t*p}Z\big) - c\frac{d}{dt}g\big(\varphi_{t*p}X,\varphi_{t*p}Y\big) \\ &\quad + g\big(\varphi_{t*p}[X,Y]_{\mathfrak{m}},\varphi_{t*p}Z\big) - g\big(\varphi_{t*p}[X,Z]_{\mathfrak{m}},\varphi_{t*p}Y\big) - g\big(\varphi_{t*p}[Y,Z]_{\mathfrak{m}},\varphi_{t*p}X\big).\end{aligned}$$

Proof. Since ξ is a geodesic vector field, $\nabla_\xi\xi = 0$. Moreover, since Y^*, $Y\in\mathfrak{m}$, is Killing, we have $g\big(\nabla_\xi Y^*,\xi\big) = 0$ and $g\big(\nabla_\xi Y^*, Z^*\big) = -g\big(\nabla_{Z^*}Y^*,\xi\big)$. Also, $g\big(\xi,\xi\big) = 1$ implies $g\big(\nabla_{X^*}\xi,\xi\big) = 0$, and $g\big(X^*,\xi\big) = 0$ implies $g\big(\nabla_{X^*}\xi, Z^*\big) = -g\big(\nabla_{X^*}Z^*,\xi\big)$. Altogether this gives

$$\begin{aligned}2g\big(\nabla_{a\xi+X^*}(b\xi+Y^*), c\xi+Z^*\big) = &-2ag\big(\nabla_{Z^*}Y^*,\xi\big) - 2bg\big(\nabla_{X^*}Z^*,\xi\big) \\ &+2cg\big(\nabla_{X^*}Y^*,\xi\big) + 2g\big(\nabla_{X^*}Y^*,Z^*\big).\end{aligned} \tag{6.4}$$

We calculate $g\big(\nabla_{X^*}Y^*,\xi\big)_{\gamma(t)}$ for X, $Y\in\mathfrak{m}$. First note that $X^*_{\gamma(t)} = X^*_{\varphi_t(p)} = \varphi_{t*p}X^*_p$ because φ_t commutes with the action of G. The fact that ξ is geodesic, X^* and Y^* are Killing vector fields, and the Levi-Civita connection is torsion-free yields

$$\begin{aligned}
\frac{d}{dt}g\big(\varphi_{t*p}X,\varphi_{t*p}Y\big) &= \frac{d}{dt}g\big(X^*_{\gamma(t)},Y^*_{\gamma(t)}\big)\\
&= g\big(\nabla_{\xi_{\gamma(t)}}X^*,Y^*_{\gamma(t)}\big)+g\big(X^*,\nabla_{\xi_{\gamma(t)}}Y^*_{\gamma(t)}\big)\\
&= -g\big(\nabla_{Y^*_{\gamma(t)}}X^*,\xi_{\gamma(t)}\big)-g\big(\nabla_{X^*_{\gamma(t)}}Y^*,\xi_{\gamma(t)}\big)\\
&= -g\big(2\nabla_{X^*_{\gamma(t)}}Y^*-[X^*,Y^*]_{\gamma(t)},\xi_{\gamma(t)}\big)\\
&= -2g\big(\nabla_{X^*_{\gamma(t)}}Y^*,\xi_{\gamma(t)}\big).
\end{aligned}$$

The last addend of (6.4) can be calculated from [Bes87, p. 7.27] as

$$g\big(\nabla_{X^*}Y^*,Z^*\big)=g\big([X^*,Y^*],Z^*\big)+g\big([X^*,Z^*],Y^*\big)+g\big([Y^*,Z^*],X^*\big).$$

Since $X^*_{\varphi_t(p)}=\varphi_{t*p}X_p$, we see that X^* is φ_t-related to itself. Thus, $[X^*,Y^*]_{\gamma(t)}=[X^*,Y^*]_{\varphi_t(p)}=\varphi_{t*p}[X^*,Y^*]_p=-\varphi_{t*p}[X,Y]_{\mathfrak{m}}$.

Substituting in (6.4) we obtain the Levi-Civita connection of M along γ. This, together with Prop. 6.5.3, finishes the proof of Prop. 6.5.4. □

Remark 6.5.5. Let $A_{\gamma(t)}$ denote the shape operator of $G\cdot\gamma(t)$ at $\gamma(t)$ with respect to the normal vector $\xi_{\gamma(t)}$, that is, $A_{\gamma(t)}(v)=-\nabla_v\xi$, $v\in T_{\gamma(t)}\big(G\cdot\gamma(t)\big)$. Then,

$$g\big(A_{\gamma(t)}X^*_{\gamma(t)},Y^*_{\gamma(t)}\big)=-g\big(\nabla_{X^*}Y^*,\xi\big)_{\gamma(t)}=\frac{1}{2}\frac{d}{dt}g\big(\varphi_{t*p}X,\varphi_{t*p}Y\big).$$

Furthermore, $X^*_{\gamma(t)}$ is a Jacobi vector field along γ with $\frac{d}{dt}_{|0}X^*_{\gamma(t)}=-A_pX^*_p$. Thus, if one is able to solve the Jacobi equation along γ, then $X^*_{\gamma(t)}$ can be calculated explicitly, and so can the shape operator $A_{\gamma(t)}$.

Definition 6.5.6. The cohomogeneity one structure S given in Prop. 6.5.4 is called *the canonical cohomogeneity one structure*.

From Prop. 6.5.3, the canonical cohomogeneity one structure satisfies

$$\tilde{T}(\xi,\,\cdot\,)=0,$$

where $\tilde{T}$ is the torsion of $\tilde{\nabla}=\nabla-S$.

Proposition 6.5.7. *Let S_1 and S_2 be two cohomogeneity one structures on (M,g) associated with ξ such that $\tilde{T}_1(\xi,\,\cdot\,)=0$ and $\tilde{T}_2(\xi,\,\cdot\,)=0$. If $\Pi_{\mathcal{T}(V)}(S_1)=\Pi_{\mathcal{T}(V)}(S_2)$, then $S_1=S_2$.*

Proof. We notice that the distribution $\mathcal{D} = \{X : g(X,\xi) = 0\}$ is independent of the choice of S_1 and S_2. Therefore, it makes sense to consider $\Pi_{\mathcal{T}(V)}(S_1) = \Pi_{\mathcal{T}(V)}(S_2)$. This means that $(S_1)_X Y = (S_2)_X Y$ for all X, $Y \in \mathcal{D}$. For $i = 1, 2$, as the Levi-Civita connection is torsion-free,

$$\tilde{T}_i(\xi, A) = -(S_i)_\xi A + (S_i)_A \xi, \quad \forall A \in TM.$$

We apply $\tilde{T}_i(\xi, \cdot) = 0$ and get $(S_i)_\xi A = (S_i)_A \xi$. For all a, $b \in \mathbb{R}$, we apply Prop. 6.3.10,

$$\begin{aligned} g((S_i)_{a\xi+X}\xi, c\xi + Y) &\overset{(4)}{=} g((S_i)_X \xi, c\xi + Y) \\ &\overset{(2)}{=} g((S_i)_X \xi, Y) \\ &\overset{(3)}{=} -\mathrm{II}(X,Y), \end{aligned}$$

which is independent of the choice of i. Then $S_1 = S_2$. □

Remark 6.5.8. The canonical cohomogeneity one structure S is the unique (up to homogeneous structure in the leaves) cohomogeneity one structure satisfying,

$$\tilde{T}(\xi, \cdot\,) = 0$$

where $\tilde{T}$ is the torsion of $\tilde{\nabla} = \nabla - S$.

6.6 Examples

In this section we examine examples and techniques to construct cohomogeneity one structures.

Parallel hyperplanes in euclidean spaces

Let $(\mathbb{R}^n, g_{\mathbb{R}^n})$ be the Euclidean space. If H is a closed subgroup of $\mathrm{SO}(n-1)$, then $G = H \ltimes \mathbb{R}^{n-1}$ acts transitively on $(\mathbb{R}^{n-1}, g_{\mathbb{R}^{n-1}})$. Consequently, G acts with cohomogeneity one on $\mathbb{R}^n$. For $p = (p_1, \dots, p_n)$, the orbit space $G \cdot p = \{(x_1, \dots x_{n-1}, p_n) : x_i \in \mathbb{R}\}$ gives a foliation without singular orbits. The unit vector field orthogonal to the leaves is $\xi_p = (0, \dots 0, 1)$ and its flow is $\varphi_t(p) = (p_1, \dots, p_n + t)$. Since φ_t is isometric and $\mathbb{R}^{n-1}$ is an abelian ideal of $\mathbb{R}^n$, the first and second rows of Prop. 6.5.4 are zero. Thus, $S = 0$.

Concentric spheres in euclidean spaces

The space $(\mathbb{R}^l - \{0\}, g_{\mathbb{R}^l})$ equipped with the Eucliden metric is isometric to the warped product $(\mathbb{R}^+ \times_f \mathbb{S}^{l-1}, g)$, $g = dr^2 + f(r)^2 g_{\mathbb{S}^{l-1}}$, where r is the distance to the origin, $f(r) = r$, and $g_{\mathbb{S}^{l-1}}$

is the round metric of the sphere. Let $\pi_2 \colon \mathbb{R}^+ \times_f \mathbb{S}^{l-1} \longrightarrow \mathbb{S}^{l-1}$ be the projection to the second coordinate. Consider a Lie group G of isometries of $(\mathbb{S}^{l-1}, g_{\mathbb{S}^{l-1}})$ and the canonical connection $\tilde{\nabla}^{\mathbb{S}^{l-1}}$ associated with it. We also consider the pseudo-group $(\mathbb{R}^+,+)$ acting transitively on $\mathbb{R}^+$. The Lie pseudo-group $\mathbb{R}^+ \times G$ acts transitively on $\mathbb{R}^+ \times_f \mathbb{S}^{l-1}$ and there is an AS-connection $\tilde{\nabla}$ (non-necessarily Riemannian and non necessarily complete) such that

$$\tilde{\nabla}\tilde{T} = 0, \qquad \tilde{\nabla}\tilde{R} = 0.$$

Actually,

$$\tilde{\nabla} = \nabla^{\mathbb{R}^+} \oplus \tilde{\nabla}^{\mathbb{S}^{l-1}},$$

where $\nabla^{\mathbb{R}^+}$ is the Levi-Civita connection for $\mathbb{R}^+$.

The Lie group G acts with cohomogeneity one on $\mathbb{R}^+ \times \mathbb{S}^{l-1} = \mathbb{R}^+ \times G/H$ and its action gives a foliation of spheres $\mathbb{S}^{l-1}(r)$ of radius $r > 0$. The normal vector field for each one of these submanifolds is, $\eta(r,p) = \partial/\partial r$ (in Euclidean coordinates, $\eta(x) = \frac{x}{||x||}$) and the associated COAS-1 connection is $\tilde{\nabla}$. We know that (see [ONe83, p. 206, Prop. 35])

$$\nabla = \nabla^{\mathbb{R}^+} \oplus \nabla^{\mathbb{S}^{l-1}} - \frac{f^2 \pi_2{}^*(g_{\mathbb{S}^{l-1}})}{f} \operatorname{grad}(f) + \sum_{i=1}^{l-1} \frac{\eta(f)}{f} (\eta^* \otimes de^i + de^i \otimes \eta^*) \frac{\partial}{\partial e^i}$$

where $\nabla^{\mathbb{S}^{l-1}}$ is the Levi-Civita connection for the sphere $\mathbb{S}^{l-1}$ and $\left\{\frac{\partial}{\partial e^1}, \ldots, \frac{\partial}{\partial e^{l-1}}\right\}$ is an orthonormal basis of $T_x\mathbb{S}^{l-1}$ and $\left\{de^1, \ldots, de^{l-1}\right\}$ is its dual basis. As $\eta(f) = 1$, we have

$$\nabla = \nabla^{\mathbb{R}^+} \oplus \nabla^{\mathbb{S}^{l-1}} - \frac{f^2 \pi_2{}^*(g_{\mathbb{S}^{l-1}})}{||x||} \eta + \frac{1}{||x||} \sum_{i=1}^{l-1} \left(\eta^* \otimes de^i + de^i \otimes \eta^*\right) \frac{\partial}{\partial e^i}.$$

Hence, the sphere foliation of $(\mathbb{R}^l - \{0\}, g)$ induced by the Lie group G possesses a cohomogeneity-one structure, represented by,

$$S = 0 \oplus S^{\mathbb{S}^{l-1}} - \frac{f^2 \pi_2{}^*(g_{\mathbb{S}^{l-1}})}{||x||} \eta + \frac{1}{||x||} \sum_{i=1}^{l-1} \left(\eta^* \otimes de^i + de^i \otimes \eta^*\right) \frac{\partial}{\partial e^i}, \tag{6.5}$$

where $S^{\mathbb{S}^{l-1}} = \nabla^{\mathbb{S}^{l-1}} - \tilde{\nabla}^{\mathbb{S}^{l-1}}$ is the homogeneous structure for the action of G on $\mathbb{S}^{l-1} = G/H$. In other words,

$$\begin{aligned} S_B C = {} & S^{\mathbb{S}^{l-1}}_{(\pi_2)_* B} (\pi_2)_* C - \frac{g((\pi_2)_* B, (\pi_2)_* C))}{r} \frac{\partial}{\partial r} \\ & + \frac{1}{r} \left(g\left(\frac{\partial}{\partial r}, B\right) (\pi_2)_* C + g\left(\frac{\partial}{\partial r}, C\right) (\pi_2)_* B \right). \end{aligned}$$

In particular, for $p \in M$, if we take $B, C \in T_p\mathbb{S}^{l-1}$, we have

$$S_B C = S_B^{\mathbb{S}^{l-1}} C - r g_{\mathbb{S}^{l-1}}(B,C)\eta$$
$$S_B \eta = \frac{1}{r}B, \quad S_\eta C = \frac{1}{r}C.$$

In a broader sense, let $(M_1, g_2, \tilde{\nabla}_1)$ and $(M_2, g_2, \tilde{\nabla}_2)$ be two Riemannian AS-manifolds, that is,

$$\begin{array}{rclrclrcl} \tilde{\nabla}_1 \tilde{R}_1 & = & 0, & \tilde{\nabla}_1 \tilde{T}_1 & = & 0, & \tilde{\nabla}_1 g_1 & = & 0 \\ \tilde{\nabla}_2 \tilde{R}_2 & = & 0, & \tilde{\nabla}_2 \tilde{T}_2 & = & 0, & \tilde{\nabla}_2 g_2 & = & 0, \end{array}$$

where $\tilde{R}_i$ and $\tilde{T}_i$ are the curvature and torsion of $\tilde{\nabla}_i$ with $i = 1, 2$. On the one hand, the product manifold $M_1 \times M_2$ is a general AS-manifold with AS-connection

$$\tilde{\nabla} = \tilde{\nabla}_1 \oplus \tilde{\nabla}_2.$$

On the other hand, for a positive function $f\colon M_1 \longrightarrow \mathbb{R}^+$, then we can consider the warped product $\left(M_1 \times_f M_2,\, g = \pi_1{}^*(g_1) + (f \circ \pi_1)^2 \pi_2{}^*(g_2)\right)$ where π_i is the projection onto M_i with $i = 1$ or 2. We write f instead of $f \circ \pi_1$. In particular, if the dimension of M_1 is 1, the warped product is a regular (locally) cohomogeneity one and the Levi-Civita connection in local coordinates is given by (again, [ONe83, p. 206, Prop. 35])

$$\nabla = \nabla_1 \oplus \nabla_2 - \frac{f^2 \pi_2{}^*(g_2)}{f}\operatorname{grad}(f) + \sum_{i=1}^{l-1} \frac{\eta(f)}{f}(\eta^* \otimes de^i + de^i \otimes \eta^*)\frac{\partial}{\partial e^i}$$

where η is a unit vector field in (M_1, g_1), $\left\{\frac{\partial}{\partial e^1}, \dots, \frac{\partial}{\partial e^{l-1}}\right\}$ is a local orthonormal basis of M_2, and $\left\{de^1, \dots, de^{l-1}\right\}$ is its dual basis. Finally, the (1, 2)-tensor

$$S = \nabla - \tilde{\nabla}$$

is a cohomogeneity one structure.

The horosphere foliation in the real hyperbolic space

The real hyperbolic space $\mathbb{R}\mathrm{H}(n)$ with the hyperbolic metric of constant scalar curvature equal to -1 is isometric to the warped product $(\mathbb{R} \times_f \mathbb{R}^{n-1}, g = dt^2 + f(t)^2 g_{\mathbb{R}^{n-1}})$, with $f(t) = e^{-t}$. Following the procedure outlined above, we can introduce the AS-connection $\tilde{\nabla}$, defined as

$$\tilde{\nabla} = \nabla^{\mathbb{R}} \oplus \nabla^{\mathbb{R}^{n-1}}$$

where $\nabla^{\mathbb{R}}$ and $\nabla^{\mathbb{R}^{n-1}}$ are the Levi-Civita connections of $\mathbb{R}$ and $\mathbb{R}^{n-1}$, respectively. Note that $\tilde{\nabla}$ corresponds to the Levi-Civita connection for the euclidean metric in $\mathbb{R}^n$. However, $\tilde{\nabla}$ is not metric with respect g, even though it satisfies

$$\tilde{\nabla}\tilde{R} = 0, \quad \tilde{\nabla}\tilde{T} = 0, \quad \tilde{\nabla}\xi = 0, \quad \tilde{\nabla}_X g = 0, \quad \tilde{T}(X,Y) \in \mathcal{L}(\xi)^{\perp},$$

where $\xi = \partial/\partial t$ and $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion of $\tilde{\nabla}$. In other words, $\tilde{\nabla}$ is an AS-connection of Thm. 6.2.2 and $S = \nabla - \tilde{\nabla}$ is a cohomogeneity one structure, where ∇ is the Levi-Civita connection of $\mathbb{R}\mathrm{H}(n)$. We now compute S explicitly.

The Levi-Civita connection for a warped product is

$$\nabla = \nabla^{\mathbb{R}} \oplus \nabla^{\mathbb{R}^{n-1}} - \frac{f^2 {\pi_2}^*(g_{\mathbb{R}^{n-1}})}{f}\mathrm{grad}(f) + \sum_{i=1}^{n-1} \frac{\xi(f)}{f}\left(\xi^* \otimes de^i + de^i \otimes \xi^*\right)\frac{\partial}{\partial e^i},$$

that is

$$\nabla = \nabla^{\mathbb{R}} \oplus \nabla^{\mathbb{R}^{n-1}} + f^2 {\pi_2}^*(g_{\mathbb{R}^{n-1}})\,\xi - \sum_{i=1}^{n-1}\left(\xi^* \otimes de^i + de^i \otimes \xi^*\right)\frac{\partial}{\partial e^i}.$$

The cohomogeneity one structure $S = \nabla - \tilde{\nabla}$ is thus

$$\begin{aligned} S &= f^2 {\pi_2}^*(g_{\mathbb{R}^{n-1}})\,\xi - \sum_{i=1}^{l-1}\left(\xi^* \otimes de^i + de^i \otimes \xi^*\right)\frac{\partial}{\partial e^i} \\ &= g((\pi_2)_*X, (\pi_2)_*Y)\xi - g(\xi,X)(\pi_2)_*Y - g(\xi,Y)(\pi_2)_*X, \end{aligned}$$

and putting $((\pi_2)_*X) = X - g(X,\xi)\xi$ and $(\pi_2)_*Y = Y - g(Y,\xi)\xi$, we finally get

$$S_X Y = g(X,Y)\xi - g(\xi,X)Y - g(\xi,Y)X + g(X,\xi)g(\xi,Y)\xi.$$

A non-canonical cohomogeneity one structure

The real hyperbolic space $\mathbb{R}\mathrm{H}(n)$, in the coordinates above, is characterized by the existence of a homogeneous structure $\bar{S}$ of linear type, see [TV83, Thm. 5.2]. This has the expression,

$$\bar{S}_X Y = g(X,Y)\xi - g(Y,\xi)X$$

where $\xi = \frac{\partial}{\partial t}$ and satisfies,

$$\tilde{\bar{\nabla}}\tilde{\bar{R}} = 0, \quad \tilde{\bar{\nabla}}\tilde{\bar{T}} = 0, \quad \tilde{\bar{\nabla}}\bar{S} = 0, \quad \tilde{\bar{\nabla}}g = 0,$$

where $\bar{S} = \nabla - \tilde{\tilde{\nabla}}$, the Levi-Civita connection is ∇, and the curvature and torsion of $\tilde{\tilde{\nabla}}$ are $\tilde{\tilde{R}}$ and $\tilde{\tilde{T}}$, respectively. We have

$$-\tilde{\tilde{T}}_A B = \bar{S}_A B - \bar{S}_B A = g(A,\xi)B - g(B,\xi)A.$$

Then, for all $X, Y \in \mathcal{D} = \{X : g(X,\xi) = 0\}$,

$$\tilde{\tilde{T}}_X Y = 0, \quad \tilde{\tilde{T}}_\xi Y = -Y \neq 0$$

which means that $\bar{S}$ is a cohomogeneity one structure, but it is not a canonical cohomogeneity one structure.

Appendix A

Expressions of Homogeneous Structures

Theorem A.1 ([AG88, Thm. 2.1] and [BGO11, Thm. 3.5])**.** *If $m \geq 6$, the space $\mathcal{K}(V)$ is decomposed into mutually orthogonal and irreducible* $\mathrm{U}(p,q)$*-submodules as*

$$\mathcal{K}(V) = \mathcal{K}_1(V) \oplus \mathcal{K}_2(V) \oplus \mathcal{K}_3(V) \oplus \mathcal{K}_4(V),$$

where

$$\begin{aligned}
\mathcal{K}_1(V) &= \Big\{ S \in \mathcal{K}(V) : S_{XYZ} = \frac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}),\ \mathrm{c}_{12}(S) = 0 \Big\}, \\
\mathcal{K}_2(V) &= \Big\{ S \in \mathcal{K}(V) : S_{XYZ} = \langle X,Y\rangle \chi_2(Z) - \langle X,Z\rangle \chi_2(Y) + \langle X,JY\rangle \chi_2(JZ) \\
&\qquad - \langle X,JZ\rangle \chi_2(JY) - 2\langle JY,Z\rangle \chi_2(JX),\ \chi_2 \in V^* \Big\}, \\
\mathcal{K}_3(V) &= \Big\{ S \in \mathcal{K}(V) : S_{XYZ} = -\frac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}),\ \mathrm{c}_{12}(S) = 0 \Big\}, \\
\mathcal{K}_4(V) &= \Big\{ S \in \mathcal{K}(V) : S_{XYZ} = \langle X,Y\rangle \chi_4(Z) - \langle X,Z\rangle \chi_4(Y) + \langle X,JY\rangle \chi_4(JZ) \\
&\qquad - \langle X,JZ\rangle \chi_4(JY) - 2\langle JY,Z\rangle \chi_4(JX),\ \chi_4 \in V^* \Big\}
\end{aligned}$$

and

$$\mathrm{c}_{12}(S)(Z) = \sum_{i=1}^{m} S_{e_i e_i Z},$$

for an orthonormal basis $\{e_1, \ldots, e_m\}$.

Theorem A.2 ([CC19, Prop. 4.2.10]). *If $m \geq 7$, the space $\mathcal{S}_+(V)$ decomposes into irreducible and mutually orthogonal* $\mathrm{U}(p,q)\times\{1\}$*-submodules as*

$$\mathcal{S}_+(V) = \mathcal{CS}_1(V)\oplus\mathcal{CS}_2(V)\oplus\mathcal{CS}_3(V)\oplus\mathcal{CS}_4(V)\oplus\mathcal{CS}_5(V)\oplus\mathcal{CS}_6(V)$$

where

$$\begin{aligned}
\mathcal{CS}_1(V) &= \Big\{S\in\mathcal{S}_+(V) : S_{XYZ} = \frac{1}{2}\left(S_{YZX}+S_{ZXY}+S_{JYJZX}+S_{JZXJY}\right), \mathrm{c}_{12}(S)=0\Big\},\\
\mathcal{CS}_2(V) &= \Big\{S\in\mathcal{S}_+(V) : S_{XYZ} = \langle X,Y\rangle\psi_2(Z)-\varepsilon\eta(X)\eta(Y)\psi_2(Z)-\langle X,Z\rangle\psi_2(Y)\\
&\qquad +\varepsilon\eta(X)\eta(Z)\psi_2(Y)+\langle X,\phi Y\rangle\psi_2(\phi Z)-\langle X,\phi Z\rangle\psi_2(\phi Y)\\
&\qquad -2\langle\phi Y,Z\rangle\psi_2(\phi X),\ \psi_2\in\hat{V}^*\Big\},\\
\mathcal{CS}_3(V) &= \Big\{S\in\mathcal{S}_+(V) : S_{XYZ} = -\frac{1}{2}\left(S_{YZX}+S_{ZXY}+S_{JYJZX}+S_{JZXJY}\right), \mathrm{c}_{12}(S)=0\Big\},\\
\mathcal{CS}_4(V) &= \Big\{S\in\mathcal{S}_+(V) : S_{XYZ} = \langle X,Y\rangle\psi_4(Z)-\varepsilon\eta(X)\eta(Y)\psi_4(Z)-\langle X,Z\rangle\psi_4(Y)\\
&\qquad +\varepsilon\eta(X)\eta(Z)\psi_4(Y)+\langle X,\phi Y\rangle\psi_4(\phi Z)-\langle X,\phi Z\rangle\psi_4(\phi Y)\\
&\qquad +2\langle\phi Y,Z\rangle\psi_4(\phi X),\ \psi_4\in\hat{V}^*\Big\},\\
\mathcal{CS}_5(V) &= \Big\{S\in\mathcal{S}_+(V) : S_{XYZ} = \alpha\eta(X)g(Y,\phi Z),\ \alpha\in\mathbb{R}\Big\},\\
\mathcal{CS}_6(V) &= \Big\{S\in\mathcal{S}_+(V) : \mathrm{c}_{2\phi 3}(\xi)=0\Big\},
\end{aligned}$$

and

$$\mathrm{c}_{12}(S)(Z) = \sum_{i=1}^{m-1} S_{e_ie_iZ},\quad \mathrm{c}_{2\phi 3}(S)(Z) = \sum_{i=1}^{m-1} S_{Ze_i\phi e_i},$$

for $\hat{V}$ the orthogonal complement to ξ, an orthonormal basis $\{e_1,\ \dots\ ,e_{m-1}\}$ of $\hat{V}$ and $\varepsilon = g(\xi,\xi)$.

Theorem A.3 ([CG90]). *If $m\geq 7$, the space $\mathcal{S}_-(V)$ decomposes into irreducible and mutually orthogonal* $\mathrm{U}(p,q)\times\{1\}$*-submodules as*

$$\mathcal{S}_-(V) = \mathcal{C}_1(V)\oplus\ \cdots\ \oplus\mathcal{C}_{12}(V)$$

where

$$
\begin{aligned}
\mathcal{C}_1(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XXY} = S_{XY\xi} = 0\Big\},\\
\mathcal{C}_2(V) &= \Big\{S \in \mathcal{S}_-(V) : \underset{X,Y,Z}{\mathfrak{S}} S_{XYZ} = 0,\, S_{XY\xi} = 0\Big\},\\
\mathcal{C}_3(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} - S_{\phi X \phi Y Z} = 0,\, \mathrm{c}_{12}(S) = 0\Big\},\\
\mathcal{C}_4(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = \langle X,Y\rangle \mu_4(Z) - \varepsilon\eta(X)\eta(Y)\mu_4(Z) - \langle X,Z\rangle \mu_4(Y)\\
&\qquad + \varepsilon\eta(X)\eta(Z)\mu_4(Y) - \langle X,\phi Y\rangle \mu_4(\phi Z) + \langle X,\phi Z\rangle \mu_4(\phi Y),\, \mu_4 \in \hat{V}^*\Big\},\\
\mathcal{C}_5(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = \beta\varepsilon\left(\eta(Y)g(X,\phi Z) - \eta(Z)g(X,\phi Y)\right),\, \beta \in \mathbb{R}\Big\},\\
\mathcal{C}_6(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = \gamma\varepsilon\left(\eta(Y)g(X,Z) - \eta(Z)g(X,Y)\right),\, \gamma \in \mathbb{R}\Big\},\\
\mathcal{C}_7(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = \eta(Z)S_{YX\xi} - \eta(Y)S_{\phi X\phi Z\xi},\, \mathrm{c}_{12}(S)(\xi) = 0\Big\},\\
\mathcal{C}_8(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = -\eta(Z)S_{YX\xi} - \eta(Y)S_{\phi X\phi Z\xi},\, \mathrm{c}_{1\phi 2}(S)(\xi) = 0\Big\},\\
\mathcal{C}_9(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = \eta(Z)S_{YX\xi} + \eta(Y)S_{\phi X\phi Z\xi}\Big\},\\
\mathcal{C}_{10}(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = -\eta(Z)S_{YX\xi} + \eta(Y)S_{\phi X\phi Z\xi}\Big\},\\
\mathcal{C}_{11}(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = -\eta(X)S_{\xi\phi Y\phi Z}\Big\},\\
\mathcal{C}_{12}(V) &= \Big\{S \in \mathcal{S}_-(V) : S_{XYZ} = \varepsilon\eta(X)\left(\eta(Y)\mu_{12}(Z) - \eta(Z)\mu_{12}(Y)\right),\, \mu_{12} \in \hat{V}^*\Big\},
\end{aligned}
$$

and

$$
\mathrm{c}_{12}(S)(Z) = \sum_{i=1}^{m-1} S_{e_i e_i Z}, \quad \mathrm{c}_{1\phi 2}(S)(Z) = \sum_{i=1}^{m-1} S_{e_i \phi e_i Z},
$$

for $\hat{V}$ the orthogonal complement to ξ, an orthonormal basis $\{e_1, \ldots, e_{m-1}\}$ of $\hat{V}$ and $\varepsilon = g(\xi,\xi)$.

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